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THE TRAJECTORY OF A PHOTON: GENERAL RELATIVITY LIGHT TIME DELAY

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The transit time of a photon or electromagnetic wave between two points in space is a measurement that is used to determine the orbits of the planets and spacecraft for the purposes of navigation and science. Both the navigation of a spacecraft and science experiments, particularly associated with General Relativity, require precise measurements of the transit time. Since the Deep Space tracking stations can measure times to within 0.1 ns or about 3 cm, it is necessary to model the transit time to this accuracy.

In this paper, a formula is developed for the total one-way transit time of a photon in the coordinate time of General Relativity. A bending term is identified that contributes about 10 ns to the transit time of a photon that grazes the Sun and is not generally included in other formula in the literature. The results are verified by performing numerical integration of the equations of motion for a photon. A numerical photon ephemeris is computed and compared with the analytic formula.

INTRODUCTION

The transit time of a photon or electromagnetic wave between two points in space is a measurement that is used to determine the orbits of the planets and spacecraft. The navigation of a spacecraft and science experiments, particularly those associated with General Relativity, require precise measurements of the transit time. Since the Deep Space Network (DSN) tracking stations can measure time with an accuracy of 0.1 ns or about 3 cm, it is necessary to model the transit time to this accuracy.

The transit time of a photon or electromagnetic wave is often referred to as the radar delay. This terminology originated with radar where a radio wave is transmitted and the delay in the reception of the reflected return is measured to determine the range. The time delay included that associated with transmission media and the path length. Individual delay terms from the troposphere, ionosphere and solar plasma are identified and used to calibrate the measured delay. For planetary spacecraft, the path length is computed from the theory of General Relativity. For a round trip travel time, the additional delay attributable to the curved space of General Relativity, over what would be computed assuming flat space, can amount to approximately 250 μs. In this memorandum, a formula is developed for the total one way transit time in the coordinate time of General Relativity. Interpretation of the transit time as a delay relative to flat space geometry is avoided because of the problem of resolving the ambiguity of flat space and curved space coordinates. Also, the determination of coordinate time avoids the problem of clocks that perform the measurement. Since coordinate time would be kept by a clock at rest with respect to the Sun and infinitely far away, the problem of reconciling measured time with coordinate time may be considered separately.

Equations for the one-way transit time associated with a photon or electromagnetic wave in a vacuum are given in Ref. 1-4. All of these references have in common a logarithmic term that is

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accurate to about 10 ns. In deriving this term, straight line motion is assumed and the contribution from the bending of the trajectory is ignored. In Refs. 2 and 3, additional terms are added to the logarithmic term that account for the error in the approximation to the metric. However, the error in omitting the bending term results in an error of about 10 ns or 3 meters which is two orders of magnitude greater than the error in the metric. In Ref. 4, the bending is accounted for and the same general approach is used for computing the time delay as in this paper. However, an attempt to verify Eq. 3.4 of Ref. 4 by numerical integration was unsuccessful. In this memorandum, an equation is developed for the one-way transit time that is accurate to better than 0.1 ns.

ANALYTIC SOLUTION FOR LIGHT TIME DELAY

The equations of motion are obtained by solution of the Einstein field equations. For a particle moving in the gravitational field of a spherical body, an exact solution may be obtained. The solution was obtained by Schwarzschild and the resulting metric tensor is defined by the line element,

$$ds^2 = g_{uv}\, dx^u dx^v \tag{1}$$

$$d\bar{s}^2 = -\left\{(1-\frac{2m}{\bar{r}})^{-1}\, d\bar{r}^2 + \bar{r}^2\, d\bar{\theta}^2 + \bar{r}^2 \sin^2\bar{\theta}\, d\bar{\phi}^2\right\} + (1-\frac{2m}{\bar{r}})\, c^2 d\bar{t}^2 \tag{2}$$

A characteristic of the Schwarzschild coordinate system is the dependence of the volume element on the coordinate $\bar{r}$. The space surrounding a spherical body is not curved but dilated as a function of $\bar{r}$ and compressed in $\bar{t}$. In order to make the relativistic system more aligned with classical coordinates, a clever coordinate transformation or change of variable was devised to make the local curved space coordinates more Euclidean. The transformed coordinate system is called isotropic Schwarzschild coordinates and the transformation is given by

$$\bar{r} = \left(1+\frac{\mu}{2c^2 r}\right)^2 r$$

$$\bar{\phi} = \phi \tag{3}$$

$$\bar{\theta} = \theta \tag{4}$$

where r, ϕ and θ are the isotropic coordinates. The isotropic form of the Schwarzschild metric is obtained by substituting the above transformations into the exact Schwarzschild equations.

$$ds^2 = \frac{\left(1-\frac{\mu}{2c^2 r}\right)^2}{\left(1+\frac{\mu}{2c^2 r}\right)^2} c^2 dt^2 - \left(1+\frac{\mu}{2c^2 r}\right)^4 \left(dr^2 + r^2 d\phi^2 + r^2 \sin^2\theta d\theta^2\right) \tag{5}$$

An analytic formula for the light time delay may be obtained by direct integration of the metric. The exact metric in isotropic Schwarzschild coordinates may be transformed from polar to Cartesian coordinates.

$$ds^2 = \frac{\left(1-\frac{\mu}{2c^2 r}\right)^2}{\left(1+\frac{\mu}{2c^2 r}\right)^2} c^2 dt^2 - \left(1+\frac{\mu}{2c^2 r}\right)^4 \left(dx^2 + dy^2 + dz^2\right) \tag{6}$$

For a photon, ds^2 is zero and the equation to be integrated for the elapsed coordinate time (t) is obtained by solving for dt.

$$dt = \frac{1}{c}\frac{\left(1+\frac{\mu}{2c^2 r}\right)^3}{\left(1-\frac{\mu}{2c^2 r}\right)} \left(dx^2 + dy^2 + dz^2\right)^{\frac{1}{2}} \tag{7}$$

Expanding in a Taylor series and retaining terms of order c^5,

$$dt = \frac{1}{c}\left(1 + 2\frac{\mu}{c^2 r} + \frac{7}{4}\frac{\mu^2}{c^4 r^2}\right)\left[dx^2 + dy^2 + dz^2\right]^{\frac{1}{2}} \tag{8}$$

The photon trajectory geometry is shown on Figure 1. The motion is constrained to the $y - z$ plane and targeted from y_1, z_1 to y_2, z_2 such that the photon arrives at the same y coordinate which is taken to be R. For this geometry, the x coordinate is zero and the y coordinate variation is much smaller than the z coordinate variation. Since $\frac{dy}{dz}$ is less than 10^{-4}, the line element differentials may be expanded as a Taylor series,

$$\left(dx^2 + dy^2 + dz^2\right)^{\frac{1}{2}} \approx dz + \frac{1}{2}\frac{dy^2}{dz} - \frac{1}{8}\frac{dy^4}{dz^3} \tag{9}$$

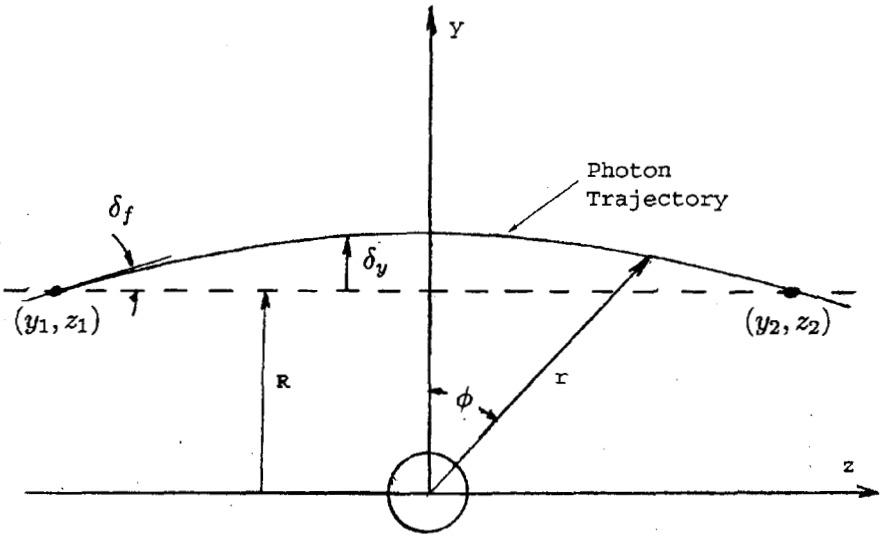

Figure 1 Photon Trajectory Geometry

Changing the y variable of integration to z and inserting Eq.(9) into Eq.(8),

$$dt \approx \frac{1}{c}\left(1 + 2\frac{\mu}{c^2 r} + \frac{7}{4}\frac{\mu^2}{c^4 r^2}\right)\left(dz + \frac{1}{2}\frac{dy^2}{dz^2}dz - \frac{1}{8}\frac{dy^4}{dz^4}dz\right) \tag{10}$$

Fully expanded, there are nine terms in Eq. (10) that are of order c^5 or greater. Consider a photon grazing the surface of the Sun. A maximum error of about 10 cm or 0.3 nanoseconds is desired. To achieve this accuracy, numerical integration of the equation of geodesics reveals that only four of the terms in Eq. (10) need be retained and these are,

$$t_2 - t_1 = \frac{1}{c}\int_{z_1}^{z_2}\left[1 + 2\frac{\mu}{c^2 r} + \frac{1}{2}\frac{dy^2}{dz^2} + \frac{7}{4}\frac{\mu^2}{c^4 r^2}\right] dz \tag{11}$$

In carrying out the integration, care should be taken in geometrically interpreting the results. A "straight line" in curved space geometry, the shortest measured distance between two points, is the photon trajectory and not the dashed line shown on Figure 1. Consider the first term of Equation 11.

$$\Delta t_f = \frac{1}{c}\int_{z_1}^{z_2} dz = \frac{1}{c}(z_2 - z_1) \tag{12}$$

This is called the flat space term. If the end points were in flat space, Δt_f would be the time a photon travels from point 1 on Figure 1 to point 2. In curved space, there is no such thing as a straight line that connects these two points. The real interpretation of this term is the mathematical result of performing the integration on the first term. The second term is called the logarithmic term for reasons that will become obvious.

$$\Delta t_{log} = \frac{1}{c}\int_{z_1}^{z_2} 2\frac{\mu}{c^2 r} dz \tag{13}$$

Integration of Eq. (13) requires an equation for r as a function of z. An iterative solution may be obtained by assuming a solution for r and integrating to obtain a first approximation for t and y as a function of z. This solution is inserted into the remainder term, the difference between the assumed and actual function, and a second iterated solution may be obtained for t and y. This method of successive approximations is continued until the required accuracy is achieved. As a starting function, "straight line" motion is assumed. Making use of the approximation that

$$r \approx \sqrt{z^2 + R^2} \tag{14}$$

$$\Delta t_{log} = 2\frac{\mu}{c^3}\int_{z_1}^{z_2} \frac{1}{\sqrt{z^2+R^2}} + \left(\frac{1}{r} - \frac{1}{\sqrt{R^2+z^2}}\right) dz \tag{15}$$

The first term of Eq. (15) integrates to the well known equation for the time delay.

$$\Delta t_{log} \approx 2\frac{\mu}{c^3}\ln\left[\frac{z_1 + (z_1^2+R^2)^{\frac{1}{2}}}{z_2 + (z_2^2+R^2)^{\frac{1}{2}}}\right] \tag{16}$$

The second term of Eq.(15), which will be referred to as the radial remainder term, requires a more accurate equation for r to be evaluated.

In order to evaluate the terms of Eq. (11) associated with bending of the trajectory, an equation for y as a function of z is needed. The y component is associated with the bending of the photon trajectory. Consider two photons in the plane of motion separated by ΔR. The plane containing these two photons and perpendicular to the velocity vector is the plane of the wave front. The bending is simply the distance one photon leads the other divided by their separation.

$$\delta = -\frac{c\Delta t_d}{\Delta R} \tag{17}$$

In the limit as ΔR approaches zero, the equation for bending is

$$\delta = -c\,\frac{dt_d}{dR} \tag{18}$$

The equation for the delay is taken to be the logarithmic term and for simplicity the bending is computed starting at closest approach ($z_2 = 0$) to the origin.

$$t_d \approx -\frac{2\mu}{c^2}\ln\left[\frac{z + (z^2+R^2)^{\frac{1}{2}}}{R}\right] \tag{19}$$

Taking the derivative,

$$\frac{dt_d}{dR} \approx -\frac{2\mu}{c^2}\left\{-\frac{1}{R} + \frac{R}{(z^2+R^2)^{\frac{1}{2}}(z+(z^2+R^2)^{\frac{1}{2}})}\right\} \tag{20}$$

Making use of the trigonometric approximations,

$$\cos\phi \approx \frac{R}{(z^2+R^2)^{\frac{1}{2}}} \tag{21}$$

$$\tan\phi \approx \frac{z}{R} \tag{22}$$

the equation for the bending reduces to

$$\delta \approx \frac{2\mu}{c^2 R}\sin\phi \tag{23}$$

The accumulated bending from z_1 to z, expressed as differentials, is given by

$$\frac{dy}{dz} \approx \delta_f + \frac{2\mu}{c^2 R}(\sin\phi - \sin\phi_1) \tag{24}$$

where δ_f is the initial angle between the photon velocity vector and the horizontal line shown on Figure 1. Referring to Figure 1, the y component of the photon is

$$y = R + \delta_y \tag{25}$$

$$\delta_y = \int_{z_1}^{z_2} \delta_f + \frac{2\mu}{c^2 R}(\sin\phi - \sin\phi_1)\ dz \tag{26}$$

$$\sin\phi \approx \frac{z}{\sqrt{R^2+z^2}} \tag{27}$$

and

$$\delta_y \approx \delta_f(z-z_1) + \frac{2\mu}{c^2 R}\sin\phi_1(z-z_1) - \frac{2\mu}{c^2 R}\left(\sqrt{z^2+R^2} - \sqrt{z_1^2+R^2}\right) \tag{28}$$

The angle δ_f may be determined by evaluating the bending over the interval from z_1 to z_2. The coordinates are rotated to target the photon to the point z_2.

$$\delta_f = \frac{\delta_y(z=z_2, \delta_f = 0)}{z_2 - z_1} \tag{29}$$

The geometrical part of the radial remainder term of Eq. (15) may be approximated by making use of Eq. (28).

$$\frac{1}{r} - \frac{1}{\sqrt{R^2+z^2}} = \frac{1}{\sqrt{(R+\delta_y)^2+R^2}} - \frac{1}{\sqrt{R^2+z^2}} \approx \frac{-R\delta_y}{(R^2+z^2)^{\frac{3}{2}}} \tag{30}$$

The complete radial remainder term (Δt_{rr}) is then given by

$$\Delta t_{rr} = \int_{z_1}^{z_2} \frac{-2\mu R}{c^3(R^2+z^2)^{\frac{3}{2}}}\left[\delta_f(z-z_1) + \frac{2\mu}{c^2 R}\sin\phi_1(z-z_1)\right.$$
$$\left. - \frac{2\mu}{c^2 R}\left(\sqrt{z^2+R^2} - \sqrt{z_1^2+R^2}\right)\right] dz \tag{31}$$

$$\Delta t_{rr} = \frac{2\mu}{c^3R^2}\left\{\frac{1}{\sqrt{R^2+z_2^2}}\left[\delta_f(R^3+z_1z_2R)+\frac{2\mu R}{c^2}\arctan\left(\frac{z_2}{R}\right)\sqrt{R^2+z_2^2}\right.\right.$$
$$\left.+\frac{2\mu}{c^2}\left(R^2\sin\phi_1+z_1z_2\sin\phi_1-z_2\sqrt{R^2+z_1^2}\right)\right]$$
$$-\frac{1}{\sqrt{R^2+z_1^2}}\left[\delta_f(R^3+z_1^2R)+\frac{2\mu R}{c^2}\arctan\left(\frac{z_1}{R}\right)\sqrt{R^2+z_1^2}\right.$$
$$\left.\left.+\frac{2\mu}{c^2}\left(R^2\sin\phi_1+z_1^2\sin\phi_1-z_1\sqrt{R^2+z_1^2}\right)\right]\right\} \tag{32}$$

The third term of Eq. (11) is the direct contribution of the trajectory bending to the time delay. This term is referred to as the bending term and is given by

$$\Delta t_b = \frac{1}{2c}\int_{z_1}^{z_2}\left(\frac{dy}{dz}\right)^2 dz \tag{33}$$

Substituting Eq. (24) for the slope into Eq. (11),

$$\Delta t_b = \frac{1}{2c}\int_{z_1}^{z_2}\left(\delta_f+\frac{2\mu}{c^2R}(\sin\phi-\sin\phi_1)\right)^2 dz \tag{34}$$

Carrying out the integration

$$\Delta t_b = \frac{1}{2c^2R}\left\{\left(\frac{4\mu^2}{c^4}+\delta_f^2R^2+\frac{4\mu\delta_fR}{c^2}\sin\phi_1+\frac{4\mu}{c^4}\sin^2\phi_1\right)[z_2-z_1]\right.$$
$$-\left(\frac{4\mu\delta_fR}{c^2}+\frac{8\mu^2}{c^4}\sin\phi_1\right)\left[\sqrt{R^2+z_2^2}-\sqrt{R^2+z_1^2}\right]$$
$$\left.+\frac{4\mu^2R}{c^4}\left[\arctan\left(\frac{R}{z_2}\right)-\arctan\left(\frac{R}{z_1}\right)\right]\right\} \tag{35}$$

The fourth and final term of Eq. (11) is the c^5 approximation to the error in the metric. This is a small term and contributes less than a nanosecond to the delay. The equation is given by

$$\Delta t_m = \frac{7}{4}\frac{\mu^2}{c^5}\int_{z_1}^{z_2}\frac{1}{r^2}dz \tag{36}$$

and may be approximated by

$$\Delta t_m \approx \frac{7}{4}\frac{\mu^2}{c^5}\int_{z_1}^{z_2}\frac{1}{R^2+z^2}\,dz \tag{37}$$

Carrying out the integration

$$\Delta t_m \approx \frac{7}{4}\frac{\mu^2}{Rc^5}\left[\arctan\left(\frac{z_2}{R}\right)-\arctan\left(\frac{z_1}{R}\right)\right] \tag{38}$$

The complete equation for the coordinate time delay of a photon moving from (y_1, z_1) to (y_2, z_2) is obtained by summing all the individual terms and

$$t_2 - t_1 \approx \Delta t_f + \Delta t_{log} + \Delta t_{rr} + \Delta t_b + \Delta t_m \tag{39}$$

Before evaluating the individual terms of Eq. (39), the parameters used in the individual terms must be determined unambiguously from the end points to the photon trajectory. If two arbitrary end ponts in the $y-z$ plane are defined by (y_1', z_1') and (y_2', z_2'), the vectors from the origin to these points are given by

$$\begin{aligned} \mathbf{r}_1 &= (0, y_1', z_1') \\ \mathbf{r}_2 &= (0, y_2', z_2') \end{aligned} \tag{40}$$

the vector from point 1 to point 2 is

$$\mathbf{r}_{12} = (0, y_2' - y_1', z_2' - y_1')$$

The angle between $\mathbf{r}_2$ and $\mathbf{r}_{12}$ is computed from the dot product

$$\theta = \arccos\left(\frac{\mathbf{r}_2 \cdot \mathbf{r}_{12}}{r_2 r_{12}}\right) \tag{41}$$

The parameters needed in Eq. (39), with the coordinates rotated as shown on Figure 1, are then given by

$$\begin{aligned} R &= r_2 \cos\theta \\ z_2 &= \sqrt{(R^2 + r_2^2)} \\ z_1 &= \sqrt{(R^2 + r_1^2)} \\ \phi_2 &= \arctan\left(\frac{z_1}{R}\right) \\ \phi_1 &= \arctan\left(\frac{z_2}{R}\right) \end{aligned} \tag{42}$$

and the angle δ_f is

$$\delta_f = \frac{1}{z_2 - z_1}\left[\frac{2\mu}{c^2 R}\sin\phi_1(z_2 - z_1) - \frac{2\mu}{c^2 R}\left(\sqrt{z_2^2 + R^2} - \sqrt{z_1^2 + R^2}\right)\right] \tag{43}$$

The fully expanded equation for the transit time is given by,

$$\begin{aligned} t_2 - t_1 \approx\ & \frac{1}{c}(z_2 - z_1) + 2\frac{\mu}{c^3}\ln\left[\frac{z_1 + (z_1^2 + R^2)^{\frac{1}{2}}}{z_2 + (z_2^2 + R^2)^{\frac{1}{2}}}\right] \\ & + \frac{2\mu}{c^3 R^2}\left\{\frac{1}{\sqrt{R^2 + z_2^2}}\left[\delta_f(R^3 + z_1 z_2 R) + \frac{2\mu R}{c^2}\arctan\left(\frac{z_2}{R}\right)\sqrt{R^2 + z_2^2}\right.\right. \\ & \left. + \frac{2\mu}{c^2}\left(R^2 \sin\phi_1 + z_1 z_2 \sin\phi_1 - z_2\sqrt{R^2 + z_1^2}\right)\right] \\ & - \frac{1}{\sqrt{R^2 + z_1^2}}\left[\delta_f(R^3 + z_1^2 R) + \frac{2\mu R}{c^2}\arctan\left(\frac{z_1}{R}\right)\sqrt{R^2 + z_1^2}\right. \end{aligned}$$

$$+\frac{2\mu}{c^2}\left(R^2\sin\phi_1+z_1^2\sin\phi_1-z_1\sqrt{R^2+z_1^2}\,\right)\Bigg]\Bigg\}$$

$$+\frac{1}{2c^2R}\Bigg\{\left(\frac{4\mu^2}{c^4}+\delta_f^2R^2+\frac{4\mu\delta_fR}{c^2}\sin\phi_1+\frac{4\mu}{c^4}\sin^2\phi_1\right)[z_2-z_1]$$

$$-\left(\frac{4\mu\delta_fR}{c^2}+\frac{8\mu^2}{c^4}\sin\phi_1\right)\left[\sqrt{R^2+z_2^2}-\sqrt{R^2+z_1^2}\right]$$

$$+\frac{4\mu^2R}{c^4}\left[\arctan\left(\frac{R}{z_2}\right)-\arctan\left(\frac{R}{z_1}\right)\right]\Bigg\}$$

$$+\frac{7}{4}\frac{\mu^2}{Rc^5}\left[\arctan\left(\frac{z_2}{R}\right)-\arctan\left(\frac{z_1}{R}\right)\right] \tag{44}$$

EQUATIONS OF MOTION

The geodesic equation describes the acceleration of a particle in space time coordinates and is given by

$$\frac{d^2x^\alpha}{ds^2}+\Gamma^\alpha_{uv}\frac{dx^u}{ds}\frac{dx^v}{ds}=0 \tag{45}$$

The equations of motion are obtained by substituting the Christoffel symbols (Γ^α_{uv}) into the geodesic equation. The Christoffel symbols are computed from the metric by,

$$\Gamma^u_{\alpha\beta}=g^{uv}\Gamma_{v\alpha\beta}$$

$$\Gamma_{u\alpha\beta}=\frac{1}{2}\left(\frac{\partial g_{u\alpha}}{\partial x^\beta}+\frac{\partial g_{u\beta}}{\partial x^\alpha}-\frac{\partial g_{\alpha\beta}}{\partial x^u}\right) \tag{46}$$

Performing the indicated tensor algebra on the metric defined by Equation 3 yields the following equations for the acceleration of a point mass in exact Schwarzschild coordinates.

$$\frac{d^2\bar{r}}{d\bar{s}^2}=\frac{m}{\bar{r}^2}\left(1-\frac{2m}{\bar{r}}\right)^{-1}\left(\frac{d\bar{r}}{d\bar{s}}\right)^2+\bar{r}\left(1-\frac{2m}{\bar{r}}\right)\left(\frac{d\bar{\phi}}{d\bar{s}}\right)^2-\frac{mc^2}{\bar{r}^2}\left(1-\frac{2m}{\bar{r}}\right)\left(\frac{d\bar{t}}{d\bar{s}}\right)^2 \tag{47}$$

$$\frac{d^2\bar{\phi}}{d\bar{s}^2}=\frac{2}{\bar{r}}\frac{d\bar{r}}{d\bar{s}}\frac{d\bar{\phi}}{d\bar{s}} \tag{48}$$

A clock carried on the particle will provide a measure of proper time ($\bar{\tau}$) defined by

$$d\bar{s}^2=c^2d\bar{\tau}^2$$

Changing the variable of integration from $\bar{s}$ to $\bar{\tau}$ and substituting the metric equation into the geodesic equation gives the following equations of motion for a point mass.

$$\frac{d^2\bar{r}}{d\bar{\tau}^2}=-\frac{\mu}{\bar{r}^2}+\left(\bar{r}-\frac{3\mu}{c^2}\right)\left(\frac{d\bar{\phi}}{d\bar{\tau}}\right)^2$$

$$\frac{d^2\bar{\phi}}{d\bar{\tau}^2}=-\frac{2}{\bar{r}}\frac{d\bar{r}}{d\bar{\tau}}\frac{d\bar{\phi}}{d\bar{\tau}}$$

$$\left(\frac{d\bar{t}}{d\bar{\tau}}\right)^2=\left(1-\frac{2m}{\bar{r}}\right)^{-1}+\frac{1}{c^2}\left(1-\frac{2m}{\bar{r}}\right)^{-2}\left(\frac{d\bar{r}}{d\bar{\tau}}\right)^2+\frac{\bar{r}^2}{c^2}\left(1-\frac{2m}{\bar{r}}\right)^{-1}\left(\frac{d\bar{\phi}}{d\bar{\tau}}\right)^2 \tag{49}$$

The trajectory of a photon differs from that of a particle or spacecraft moving at the speed of light even in the limit of very small mass. The difference arises because a photon has zero rest mass and thus there is no force of gravity acting on the photon that gives rise to Newtonian acceleration. The photon follows the contour of curved space and the resulting path is the so called null geodesic. An observer's clock on the photon will not register any passage of time as the photon moves from point (y_1, z_1) to point (y_2, z_2) resulting in ds being zero. The problem of ds approaching zero in the geodesic equation is resolved by introducing the affine parameter τ that acts like a clock on the photon. Only this clock runs at a rate determined by the local gravity field strength and τ is referred to as proper time. The proper time associated with a photon is simply the integral of these rates along the path of the photon. The difference in the proper time between two points times the speed of light is the distance that one would measure with a meter stick along the path of the photon. Since the measured time on the photon is zero, the metric degenerates to indeterminate forms that must be evaluated in the limit as ds approaches zero. The indeterminate form ds/ds, which has the value of 1 for a spacecraft, has the value 0 for a photon in the limit as ds approaches zero. The equations of motion for a photon are thus given by,

$$\frac{d^2\bar{r}}{d\bar{\tau}^2} = \left(\bar{r} - \frac{3\mu}{c^2}\right)\left(\frac{d\bar{\phi}}{d\bar{\tau}}\right)^2$$

$$\frac{d^2\bar{\phi}}{d\bar{\tau}^2} = \frac{2}{\bar{r}}\frac{d\bar{r}}{d\bar{\tau}}\frac{d\bar{\phi}}{d\bar{\tau}}$$

$$\left(\frac{d\bar{t}}{d\bar{\tau}}\right)^2 = \frac{1}{c^2}\left(1 - \frac{2m}{\bar{r}}\right)^{-2}\left(\frac{d\bar{r}}{d\bar{\tau}}\right)^2 + \frac{\bar{r}^2}{c^2}\left(1 - \frac{2m}{\bar{r}}\right)^{-1}\left(\frac{d\bar{\phi}}{d\bar{\tau}}\right)^2 \tag{50}$$

Isotropic Schwarzschild Coordinates

The exact isotropic Schwarzschild equations of motion may be obtained by substituting the coordinate transformation given by Equation 4 into the exact Schwarzschild equations of motion given by Equation 45 for a point mass and Equation 46 for a massless particle or photon.

$$\frac{d^2r}{d\tau^2} = -\eta\,\frac{\mu}{r^2}\left(1 + \frac{\mu}{2c^2r}\right)^{-4} + \left(1 - \frac{\mu^2}{4c^4r^2}\right)^{-1}$$
$$\times\left\{\frac{\mu^3}{2c^4r^5}\left(1 + \frac{\mu}{2c^2r}\right)^{-4}\left(\frac{dr}{d\tau}\right)^2 + \left[\left(1 + \frac{\mu}{2c^2r}\right)^2 r - \frac{3\mu}{c^2}\right]\left(\frac{d\phi}{d\tau}\right)^2\right\}$$

$$\frac{d^2\bar{\phi}}{d\tau^2} = -\frac{\left(1 - \frac{\mu}{2c^2r}\right)}{\left(1 + \frac{\mu}{2c^2r}\right)}\frac{2}{r}\frac{dr}{d\tau}\frac{d\bar{\phi}}{d\tau}$$

$$\left(\frac{dt}{d\tau}\right)^2 = \eta\,\frac{\left(1 + \frac{\mu}{2c^2r}\right)^2}{\left(1 - \frac{\mu}{2c^2r}\right)^2} + \frac{1}{c^2}\frac{\left(1 + \frac{\mu}{2c^2r}\right)^6}{\left(1 - \frac{\mu}{2c^2r}\right)^2}\left[\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\bar{\phi}}{d\tau}\right)^2\right] \tag{51}$$

The variable η is used to separate the point mass equations of motion from the photon equations of motion. For a point mass, η is equal to one and for a photon, η is equal to zero. Retaining terms to order c^2, the exact isotropic Schwarzschild equations of motion for a point mass and a photon may be approximated by

$$\frac{d^2r}{d\tau^2} \approx -\eta\frac{\mu}{r^2}\left(1 - \frac{2\mu}{c^2r}\right) + \left(r - \frac{2\mu}{c^2}\right)\left(\frac{d\phi}{d\tau}\right)^2$$

$$\frac{d^2\bar{\phi}}{d\tau^2} \approx -\left(1-\frac{\mu}{c^2 r}\right)\frac{2}{r}\frac{dr}{d\tau}\frac{d\bar{\phi}}{d\tau}$$

$$\left(\frac{dt}{d\tau}\right)^2 \approx \eta\left(1+\frac{2\mu}{c^2 r}\right)+\frac{1}{c^2}\left(1+\frac{4\mu}{c^2 r}\right)\left[\left(\frac{dr}{d\tau}\right)^2+r^2\left(\frac{d\bar{\phi}}{d\tau}\right)^2\right] \tag{52}$$

The approximate isotropic Schwarzschild metric may be obtained by discarding terms higher than order c^2 from Eq. (5).

$$ds^2 \approx \left(1-\frac{2\mu}{c^2 r}+\frac{2\mu^2}{c^4 r^2}\right)c^2 dt^2-\left(1+\frac{2\mu}{c^2 r}\right)\left(dr^2+r^2 d\phi^2\right) \tag{53}$$

Solving the approximate metric for t, an equation similar to Eq. (10) is obtained. The only difference is the coefficient for the last term associated with the error in the metric. This term has a coefficient of -1 instead of $\frac{7}{4}$. When carried forward to Eq. (40) for the time delay, the last two terms also have a coefficient of -1 instead of $\frac{7}{4}$. The difference in the rendering of these terms provides an indication of the error associated with the approximate form of the isotropic metric since Equation 40 is accurate to better than order c^2. The error in the time delay (ϵ_t) associated with the approximate isotropic Schwarzschild form of the metric is thus given by

$$\epsilon_t = \frac{11}{4}\frac{\mu^2}{Rc^5}\left[\arctan\left(\frac{z_2}{R}\right)-\arctan\left(\frac{z_1}{R}\right)\right] \tag{54}$$

For a photon that traverses the solar system and grazes the surface of the Sun, the error attributable to the metric approximation is

$$\epsilon_{graze} = \frac{11\pi\mu^2}{4Rc^5} \tag{55}$$

or 0.089 ns (2.67 cm). Since the approximate form of the isotropic Schwarzschild metric is the basis for the N-body equations of motion, the error in the time delay should be consistent with the error in determining the mass of the bodies in the solar system and the position of the end points. The c^4 term is sometimes included in the t coefficient of the approximate isotropic metric yielding higher precision for the metric. However, the benefit of a high precision time measurement may be lost if the end points are determined from a lower approximation.

COMPARISON OF ANALYTIC FORMULA WITH NUMERICAL INTEGRATION

Equation 40 for the time delay associated with a photon or electromagnetic wave is given as a function of parameters that may be computed from the exact isotropic coordinates of the end points. This equation is exact in the sense that an equation in Schwarzschild curved space coordinates could be obtained by back substitution of the coordinate transformations defining spherical coordinates and isotropic Schwarzschild coordinates that is accurate to the error of the terms omitted from the expansion. These error terms arise from two sources. The first and most obvious is the error in the metric. The metric error terms have been accounted for to order c^4. The second source of error is in discarding remainder terms associated with the expansion. The major terms have been accounted for but the error in describing the trajectory bending to c^4 and higher will ultimately limit the accuracy of Equation 40. Only the c^2 bending is included and since this term contributes only a minor correction on the order of 10 ns, the c^4 term would contribute several orders of magnitude less than the c^2 term.

In order to determine the veracity of Equation 40, a comparison with the time delay computed from integration of the geodesic equations of motion was made and the result plotted on Figure 2. In carrying out the numerical integration, a photon was initialized with a z coordinate of -149,000,000 km and y coordinate of 696,000 km. The y component of velocity was set to zero and the z component

to c. The x coordinates of position and velocity were set to zero. Thus the photon is initialized with a velocity magnitude equal to the speed of light and parallel to the z axis about 1 a.u. from the sun and on a flight path that would graze the surface of the Sun if there were no bending due to General Relativity. The polar coordinates of the initial conditions were used to initialize the equations of motion (Equation 51) and these were integrated by a fourth order Runge Kutta integrator with fifth order error control. The integration was stopped at various times along the flight path and Equation 40 was evaluated. In evaluating Equation 40, the required parameters were computed from the integrated initial coordinates and the integrated coordinates at the time of the evaluation. This required a rotation of coordinates to place the coordinate axes into the alignment shown on Figure 1 before the parameters of Equation 40 were computed.

Also shown on Figure 2 are some of the individual groupings of terms from Equation 40. The linear term has been omitted since this term would require an additional 6 cycles of logarithmic scale. The dashed curve is the difference between the time delay computed by Equation 40 and the results of numerical integration. This difference is attributed to error in the numerical integration algorithm. This was verified by setting the mass of the Sun to zero and integrating straight line motion in the same coordinate system. Furthermore, it would be difficult to imagine real error terms with the periodicity shown on Figure 2. Unfortunately, the integration error masked the error in the metric. Therefore, Equation 40 could only be verified to about 0.1 ns which is about the same level of error as the error in the metric. The integration error of about 10^{-14} t is consistent with the error obtained integrating spacecraft orbits for navigation.

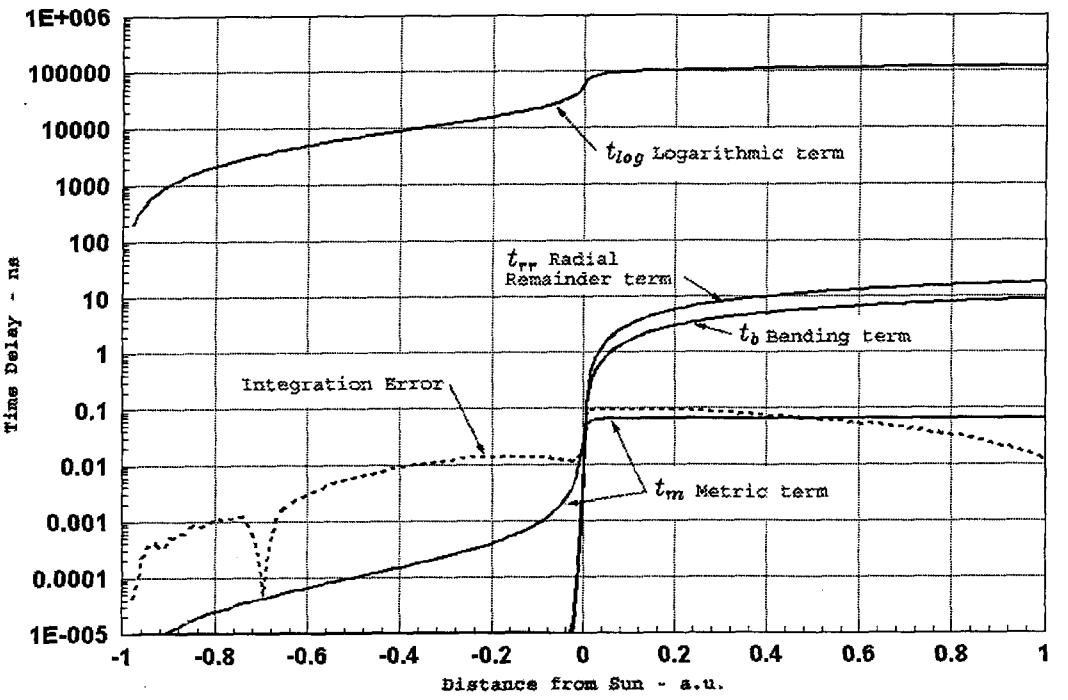

Figure 2 Time Delay for Solar Graze

Observe that the radial remainder term and bending term cause errors on the order of 10 ns or 37 cm. For relativity experiments carried out with laser beams between spacecraft, this error would be too large to be ignored. Even with conventional radio metric measurements, measurement errors of 10 cm are common. In any event, it does not make sense to include corrections for the error in the metric and the logarithmic term in the time delay and exclude the bending and radial remainder

term.

PHOTON EPHEMERIS

An ephemeris is by definition the tabulation of the positions of a celestial body as a function of time. If a photon is regarded as a celestial body, the position may be tabulated making use of Equation 40 for the time and Equation 27 for the y coordinate of position. The x position is zero and the z coordinate is the independent parameter. This ephemeris may be tabulated along with the results from numerical integration of the equation of geodesics for a photon (Equation 45 or Equation 49). Since the independent parameter for the equation of geodesics is proper time, the following procedure was used to make the comparison. First, a trajectory was integrated from the initial conditions defined above for the generation of Figure 2. The initial conditions are the first record tabulated in Table 1. The velocity terms have been omitted since the speed of light is constant with respect to τ, the measured or proper time. The equation of geodesics for a photon only turn the velocity vector and do not change the magnitude. A first approximation to the direction of the velocity vector is given by Equation 23. For the geometry selected here, the angle δ_f is taken to be zero. The second line of each record gives the value of the coordinates computed from the analytic formulae. The z coordinates used in the analytic formula were obtained by transformation of the end points obtained from the numerical integration result to bring the geometry into the alignment shown on Figure 1.

The geodesic equations of motion that are integrated may be in the classical Schwarzschild coordinates of Equation 46 or the exact isotropic Schwarzschild coordinates of Equation 50. Either set of equations will yield the same exact result. If the classical Schwarzschild equations are integrated,

Table 1
Photon Ephemeris

PROPER TIME	Z	Y	COORDINATE TIME (t)
0.0000000000000000E+00	-0.1490000000000000E+09	0.6960000000000000E+06	0.0000000000000000E+00
	ANALYTIC FORMULA	0.6960000000000000E+06	0.0000000000000000E+00
0.1000000000000000E+03	-0.1190207542000003E+09	0.6959999996503065E+06	0.1000000022129771E+03
	ANALYTIC FORMULA	0.6959999996504503E+06	0.1000000022129760E+03
0.4000000000000000E+03	-0.2908301679999592E+08	0.6959999771158417E+06	0.4000000160930987E+03
	ANALYTIC FORMULA	0.6959999771158228E+06	0.4000000160931119E+03
0.4900000000000000E+03	-0.2101695579995966E+07	0.6959995374160476E+06	0.4900000417174753E+03
	ANALYTIC FORMULA	0.6959995374155714E+06	0.4900000417174866E+03
0.4950000000000000E+03	-0.6027332899947603E+06	0.6959986645459520E+06	0.4950000519750542E+03
	ANALYTIC FORMULA	0.6959986645457718E+06	0.4950000519750684E+03
0.5000000000000000E+03	0.8962289999880246E+06	0.6959913960385144E+06	0.5000000702418170E+03
	ANALYTIC FORMULA	0.6959913959812428E+06	0.5000000702418982E+03
0.5050000000000000E+03	0.2395191289940149E+07	0.6959792669838322E+06	0.5050000788965964E+03
	ANALYTIC FORMULA	0.6959792665023379E+06	0.5050000788966800E+03
0.6000000000000000E+03	0.3087547479891812E+08	0.6957379602255528E+06	0.6000001038802436E+03
	ANALYTIC FORMULA	0.6957378801101440E+06	0.6000001038803274E+03
0.1000000000000000E+04	0.1507924579945738E+09	0.6947203283209250E+06	0.1000000119502137E+04
	ANALYTIC FORMULA	0.6947191658534410E+06	0.1000000119502140E+04

C=0.2997924580000000E+06 GM=0.1327124399000000E+12

the initial conditions must be transformed using Equation 4 and the results of the integration for $\bar{r}$ transformed back to isotropic coordinates for comparison. Integration of the approximate isotropic equations of motion, that are the basis for the n-body equations of motion of De Sitter and Einstein, yields a result that differs from the exact equations by the error given by Equation 53.

The results shown in Table 1 are tabulated for a photon that crosses the orbit of the Earth and grazes the Sun. The coordinate time is displayed to 16 digits which is about the machine precision

of the computer used for the computation, a PC with 53 bit mantissa and 11 bit exponent. The comparison is accurate to 14 decimal places and the error may be attributed to integrator round off.

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