Rigid Body Equations of Motion for Modeling and Control of Spacecraft Formations - Part 1: Absolute Equations of Motion.

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Abstract

In this paper, we present a tensorial (i.e., coordinate-free) derivation of the equations of motion of a formation consisting of N spacecraft each modeled as a rigid body. Specifically, using spatial velocities and spatial forces we demonstrate that the equations of motion for a single free rigid body (i.e., a single spacecraft) can be naturally expressed in four fundamental forms. The four forms of the dynamic equations include (1) motion about the system center-of-mass in terms of absolute rates-of-change, (2) motion about the system center-of-mass in terms of body rates of change, (3) motion about an arbitrary point fixed on the rigid body in terms of absolute rates-of-change, and (4) motion about an arbitrary point fixed on the rigid body in terms of body rates-of-change. We then introduce the spatial Coriolis dyadic and discuss how a proper choice of this non-unique tensor leads to dynamic models of formations satisfying the skew-symmetry property required by an important class of nonlinear tracking control laws. Next, we demonstrate that the equations of motion of the entire formation have the same structure as the equations of motion of an individual spacecraft. The results presented in this paper form the cornerstone of a coordinate-free modeling environment for developing dynamic models for various formation flying applications.

1 Introduction

The ability to accurately capture the dynamic behavior of separated spacecraft formations in both deep space and in orbit around a central body is critical to the success of many planned and future NASA missions. For example, the development and assessment of high-precision formation flying control laws will require a spectrum of spacecraft dynamic models ranging from point mass models to multi-flexible body models.

To this end, we develop a tensorial formulation of the equations of motion of formations consisting of N separated spacecraft. In addition, by utilizing the concept of a spatial vector (viz., a vectrix consisting of both rotational and translational vector quantities) it is possible to unify formation translational and rotational dynamics into a single framework. The coordinate free approach using spatial vectors allows one to have maximal physical insight into the structure of formation dynamics with a minimum of notational overhead. The coordinate-free approach discussed here is based on the use of direct tensor notation to formulate the equations of motion of the system. This approach is especially powerful in applications where a large number of observers (i.e., reference frames) are involved in the dynamic analysis. Further, once a specific set of generalized coordinates has been chosen, the tensorial equations admit a concise matrix form which is amenable to computer simulation. The coordinate-free modeling architecture developed in this paper also facilitates the design of nonlinear tracking control laws for separated spacecraft formations.

The primary focus of this paper is on the application of the coordinate-free approach to develop the equations of motion for formations consisting of N spacecraft, where each spacecraft is modeled as a single rigid body. Much of the research done to date in the area of formation flying dynamics has concentrated on the development of 3 degree-of-freedom (3DOF) translational equations of motion along with associated disturbance models. For example, the linearized translational motion equations of one spacecraft relative to another spacecraft in a circular orbit (commonly called the Clohessy-Wiltshire-Hill equations) has been addressed by many researchers; e.g., see [10]. The assumption of a circular reference orbit has been relaxed in a number of papers; e.g., see [1] for an overview. Using coordinate-free notation, [11] discusses the relative translational dynamics of formations in deep space and provides insight into the validity of utilizing linear dynamic models ("double integrator models") for control law design. A unified 6 DOF description of formation flying dynamics (as well as guidance and control) has been elusive; notable exceptions are [8] and [14].

The rest of this paper is organized as follows. First, we discuss material from rigid body kinematics and tensor analysis that are needed in the sequel. Then, using the concept of spatial velocities and forces (i.e., combining linear/angular velocities and forces/torques into a single entity), we demonstrate that the equations of motion of a rigid body can be naturally expressed in four distinct forms. Although each form is an exact description of the dynamics of a single rigid body, each form is not equivalent for developing dynamic models and control laws for separated spacecraft formations. To this end, we then discuss the proper selection of the spatial Coriolis dyadic, which is required in an important class of adaptive control laws used for the control of systems of bodies including underwater vehicles [2], flexible space structures [7], and robotic systems [12]. Next, we demonstrate that the absolute equations of motion of the N individual spacecraft can be concatenated to form the motion equations of the entire formation. Moreover, it is shown that the equations of motion at the formation-level have the same structure as the equations of motion of a single rigid spacecraft. The final form of the absolute equations of motion provide the first step toward a complete description of the dynamics of formations and can be tailored to applications in various dynamic environments.

2 Differentiation of Vectors in Rotating Frames of Reference

In this section we discuss background material from rigid body kinematics and tensor analysis that is required in the sequel; see [4] and [6] for further information. Consider a geometric vector $\vec{Q}$ describing some physical quantity (e.g. velocity, force, angular momentum, etc.) of interest. Here we are making the important distinction between *geometric* or *Gibbsian* vectors and column matrices or 3-tuples of real numbers. A geometric vector $\vec{Q}$ is a quantity possessing magnitude, direction, and obeying the parallelogram law of addition in three dimensional Euclidean point space, denoted E^3. A geometric vector should be thought of as an arrow or directed line segment in E^3. In particular, a vector $\vec{Q}$ is a geometric object that exists independently of any particular basis chosen for E^3, while a column matrix of numbers $Q = [Q_1, Q_2, Q_3]^T \in \Re^{3\times 1}$ is the representation of a geometric vector in a particular basis. It follows that a column matrix Q depends on *both* the underlying vector and the particular frame of reference.

In rigid body kinematics we limit our attention to a special class of basis vectors for E^3 called reference frames. A reference frame consists of a right-handed set of three mutually orthonormal vectors located at an arbitrary point (called the origin of the frame) in E^3. The basis vectors associated with a reference frame can be easily accounted by defining a *vectrix* [1] as follows: $\vec{\mathcal{F}}_A \triangleq [\vec{a}_1\ \vec{a}_2\ \vec{a}_3]^T$. In other words, the elements of the vectrix $\vec{\mathcal{F}}_A$ are the basis vectors characterizing the given frame of reference. In the sequel, a reference frame will be denoted as $\mathcal{F}_A$ and the vectrix associated with the frame as $\vec{\mathcal{F}}_A$.

Once a reference frame $\mathcal{F}_A$ has been defined, a geometric vector can be represented uniquely as $\vec{Q} = Q_1\vec{a}_1 + Q_2\vec{a}_2 + Q_3\vec{a}_3$ or equivalently as $\vec{Q} = \vec{\mathcal{F}}_A^T Q_A$. The real numbers $Q_i = \vec{Q}\cdot\vec{a}_i$ are called the components of $\vec{Q}$ relative to $\mathcal{F}_A$. In short, Q_A is the column matrix whose entries are the components of $\vec{Q}$ in $\mathcal{F}_A$. Note that when a different reference frame $\mathcal{F}_B$ is chosen, the same underlying geometric object $\vec{Q}$ will admit a different column matrix representation $Q_B = [Q_1', Q_2', Q_3']^T \in \Re^{3\times 1}$.

Physically, a reference frame can be identified with an *observer* who is rigidly mounted along the three mutually orthogonal axes of $\mathcal{F}_A$. In this paper, all observers are assumed to measure the same absolute time irrespective of their state of motion.

A fundamental result that will be used in the sequel is the *Transport Theorem* for geometric vectors [6]:

$$\overset{A}{\vec{Q}} = \overset{B}{\vec{Q}} + \left[{}^A\vec{\omega}^B\right]\vec{Q} \tag{1}$$

Here $\vec{Q}$ denotes an arbitrary geometric vector, $[\vec{a}]\vec{b} = \vec{a}\times\vec{b}$, ${}^A\vec{\omega}^B$ denotes the angular velocity of $\mathcal{F}_B$ in $\mathcal{F}_A$, $\overset{A}{\vec{Q}} \triangleq \dot{Q}_1\vec{a}_1 + \dot{Q}_2\vec{a}_2 + \dot{Q}_3\vec{a}_3$, and $\overset{B}{\vec{Q}} \triangleq \dot{Q}_1'\vec{b}_1 + \dot{Q}_2'\vec{b}_2 + \dot{Q}_3'\vec{b}_3$. The term $\overset{A}{\vec{Q}}$ (resp. $\overset{B}{\vec{Q}}$) can be interpreted physically as the rate of change of $\vec{Q}$ as seen by an observer rigidly mounted to the axes of $\mathcal{F}_A$ (resp. $\mathcal{F}_B$). As a consequence, if $\vec{Q}$ is a vector fixed in $\mathcal{F}_A$ (resp. $\mathcal{F}_B$) then $\overset{A}{\vec{Q}} = \vec{0}$ (resp. $\overset{B}{\vec{Q}} = \vec{0}$).

In the sequel we will also consider tensors of second rank, called *dyadics*. In complete analogy with a vector, a dyadic $\overleftrightarrow{T}$ is a geometric object that is independent of any observer. For our purposes, we regard a dyadic as a linear operator $\overleftrightarrow{T}: E^3 \mapsto E^3$; i.e., a dyadic is a linear mapping on the space of geometric vectors. However, once a reference frame $\mathcal{F}_A$ has been introduced, a dyadic can be represented uniquely as $\overleftrightarrow{T} = \vec{\mathcal{F}}_A^T T_A \vec{\mathcal{F}}_A$ where the elements of the 3×3 matrix T_A are called the components (or matrix representation) of $\overleftrightarrow{T}$ relative to $\mathcal{F}_A$. In short, T_A is the 3×3 matrix whose entries T_{ij} are the representation of $\overleftrightarrow{T}$ in $\mathcal{F}_A$. Note that when a different reference frame $\mathcal{F}_B$ is chosen, the same underlying geometric object $\overleftrightarrow{T}$ will admit a different matrix representation, given by a different 3×3 matrix $T_B \in \Re^{3\times 3}$ with entries T_{ij}'.

[1] Although this notational device apparently goes back to the 1960's, the phrase vectrix (i.e., part vector, part matrix) was popularized by Hughes in [4]. Specifically, a vectrix is a matrix whose elements are geometric vectors.

Table 1: Four Forms of the Dynamic Equations

Form	Reference Point	Vector Derivatives
Form 1	Center-of-Mass	Inertial
Form 2	Center-of-Mass	Body
Form 3	Arbitrary	Inertial
Form 4	Arbitrary	Body

The following generalization of the Transport Theorem for dyadics will also be used in the sequel:

$$\overset{A}{\overset{\leftrightarrow}{T}} = \overset{B}{\overset{\leftrightarrow}{T}} + \left[{}^A\vec{\omega}^B\right] \overset{\leftrightarrow}{T} - \overset{\leftrightarrow}{T} \left[{}^A\vec{\omega}^B\right] \tag{2}$$

Here $\overset{\leftrightarrow}{T}$ denotes an arbitrary second rank tensor, $\overset{A}{\overset{\leftrightarrow}{T}} \triangleq \dot{T}_{ij}\vec{a}_i\vec{a}_j$, $\overset{B}{\overset{\leftrightarrow}{T}} \triangleq \dot{T}'_{ij}\vec{b}_i\vec{b}_j$ and the notation $\vec{a}_i\vec{a}_j$ (resp. $\vec{b}_i\vec{b}_j$) denotes the dyadic (or tensor) product. Note that we have employed the summation convention in the above expressions. The reader should consult [5] for further information.

Spacecraft Equations of Motion: Rigid Body Models

In this section we show that the equations of motion of a spacecraft modeled as a single rigid body can be naturally expressed in four different forms. To obtain maximum insight into the structure of the equations of motion, coordinate-free vector/dyadic notation will be utilized throughout. The four forms of the equations of motion are classified as follows: (1) motion equations about the system center-of-mass in terms of absolute rates-of-change, (2) motion equations about the system center-of-mass in terms of body rates of change, (3) motion equations about an arbitrary point fixed on the rigid body in terms of absolute rates-of-change, and (4) motion equations about an arbitrary point fixed on the rigid body in terms of body rates-of-change. The four forms[2] of the equations of motion are summarized in Table 1.

Euler's Fundamental Laws of Mechanics

The following *independent* laws of mechanics, due to Euler in 1775, characterize the momentum balance of a single rigid body[3]:

$$\overset{N}{\vec{h}_c} = \vec{\tau}_c \tag{3}$$

$$\overset{N}{\vec{p}_c} = \vec{f} \tag{4}$$

where $\vec{p}_c$ is the absolute linear momentum of the body, $\vec{h}_c$ is the absolute angular momentum about the mass center of the system, $\vec{f}$ is the resultant external force acting on the body, $\vec{\tau}_c$ is the resultant torque about the system center-of-mass, and $\overset{N}{(\bullet)}$ denotes the rate-of-change relative to an inertial frame (i.e., an inertially fixed observer). Equation (3) is called the *balance of angular momentum* and (4) is called the *balance of linear momentum.* We now define the *spatial momentum* and *spatial force* vectors[4] as follows:

$$\vec{H}_c \triangleq \begin{bmatrix} \vec{h}_c \\ \vec{p}_c \end{bmatrix} \tag{5}$$

$$\vec{F}_c \triangleq \begin{bmatrix} \vec{\tau}_c \\ \vec{f} \end{bmatrix} \tag{6}$$

As a result, Euler's Laws of Mechanics (3)-(4) can be expressed in the concise form

$$\overset{N}{\vec{H}_c} = \vec{F}_c \tag{7}$$

The use of spatial vectors (i.e., the combination of linear and angular quantities) not only leads to a simplified set of motion equations and deeper insight into the dynamic behavior of rigid bodies, but also *allows the unification of translational and rotational motion within a single framework.* The spatial momentum vector (about the center-of-mass) of a rigid body is related to the spatial velocity as follows

$$\vec{H}_c = \overset{\leftrightarrow}{M}_c \vec{V}_c \tag{8}$$

where the *spatial velocity* is defined as

$$\vec{V}_c \triangleq \begin{bmatrix} \vec{\omega} \\ \vec{v}_c \end{bmatrix} \tag{9}$$

and the *spatial inertia dyadic*[5] is

$$\overset{\leftrightarrow}{M}_c = \begin{bmatrix} \overset{\leftrightarrow}{J}_c & \overset{\leftrightarrow}{0} \\ \overset{\leftrightarrow}{0} & m\overset{\leftrightarrow}{1} \end{bmatrix} \tag{10}$$

Here m denotes the (assumed constant) mass of the body, $\vec{v}_c$ denotes the absolute velocity of the center-of-mass of the body, $\overset{\leftrightarrow}{J}_c$ denotes the inertia dyadic of the body about its center of mass, $\overset{\leftrightarrow}{1}$ is the unit dyadic, $\overset{\leftrightarrow}{0}$ is the null dyadic, and $\vec{\omega} = {}^N\vec{\omega}^B$ is the angular velocity of the body in $\mathcal{F}_N$. Note that the spatial inertia dyadic $\overset{\leftrightarrow}{M}_c$ is symmetric.

[2]Note that other forms of the equations of motion result when inertial derivatives are expressed with respect to an obsever having *arbitrary motion* relative to the body. See Greenwood [3] for further details.

[3]More generally, Euler's laws can be used to describe the motion of a finite, arbitrarily deforming body; see [5] and [13] for further information.

[4]Note that the spatial velocity, spatial momentum, and spatial force are vectrices.

[5]A spatial dyadic is a matrix of dyadics.

First (Fundamental) Form of the Equations of Motion

In this section we obtain the equations of motion of a single rigid body about the center-of-mass in terms of absolute derivatives.

Substituting the expression for the spatial momentum (8) into the momentum balance (7) and performing the inertial derivative we find

$$\overleftrightarrow{M}_c \overset{N}{\vec{V}}_c + \overset{N}{\overleftrightarrow{M}}_c \vec{V}_c = \vec{F}_c \tag{11}$$

In order to determine $\overset{N}{\overleftrightarrow{M}}_c$ we generalize (2) for use with *spatial* dyadics:

$$\overset{N}{\overleftrightarrow{M}}_c = \overset{B}{\overleftrightarrow{M}}_c + \left[\vec{\Omega}\right] \overleftrightarrow{M}_c - \overleftrightarrow{M}_c \left[\vec{\Omega}\right] \tag{12}$$

where

$$\left[\vec{\Omega}\right] \triangleq \begin{bmatrix} [\vec{\omega}] & \overleftrightarrow{0} \\ \overleftrightarrow{0} & [\vec{\omega}] \end{bmatrix} \tag{13}$$

Noting that $\overset{B}{\overleftrightarrow{M}}_c = \overleftrightarrow{0}$ for a rigid body of constant mass, we find that the equations of motion are

$$\overleftrightarrow{M}_c \overset{N}{\vec{V}}_c + \overleftrightarrow{C}_1 \vec{V}_c = \vec{F}_c \tag{14}$$

where

$$\begin{aligned} \overleftrightarrow{C}_1 &= \overset{N}{\overleftrightarrow{M}}_c && (15) \\ &= \left[\vec{\Omega}\right] \overleftrightarrow{M}_c - \overleftrightarrow{M}_c \left[\vec{\Omega}\right] && (16) \end{aligned}$$

Expanding out (14) we find

$$\begin{bmatrix} \overleftrightarrow{J}_c & \overleftrightarrow{0} \\ \overleftrightarrow{0} & m\overleftrightarrow{1} \end{bmatrix} \begin{bmatrix} \overset{N}{\vec{\omega}} \\ \overset{N}{\vec{v}}_c \end{bmatrix} + \begin{bmatrix} [\vec{\omega}]\overleftrightarrow{J}_c \vec{\omega} \\ \vec{0} \end{bmatrix} = \begin{bmatrix} \vec{\tau}_c \\ \vec{f} \end{bmatrix} \tag{17}$$

Equation (17) (resp. (14)) will be called the *fundamental form* of the equations of motion for a single rigid body.

Second Form of the Equations of Motion

In this section we develop the equations of motion about the center-of-mass in terms of body rates-of-change.

Applying the transport formula (1) to the vectors $\vec{\omega}$ and $\vec{v}_c$ we find $\overset{N}{\vec{\omega}} = \overset{B}{\vec{\omega}}$ and $\overset{N}{\vec{v}}_c = \overset{B}{\vec{v}}_c + [\vec{\omega}]\vec{v}_c$. Note that in terms of spatial velocities the above equations can be written as

$$\overset{N}{\vec{V}}_c = \overset{B}{\vec{V}}_c + \left[\vec{\Omega}\right] \vec{V}_c \tag{18}$$

where $\left[\vec{\Omega}\right]$ is as defined in (13). Also $[\vec{\omega}]\vec{\omega} = \vec{0}$ has been used in (18).

Substituting (18) into (14) results in

$$\overleftrightarrow{M}_c \left(\overset{B}{\vec{V}}_c + \left[\vec{\Omega}\right] \vec{V}_c\right) + \overleftrightarrow{C}_1 \vec{V}_c = \vec{F}_c \tag{19}$$

Rearranging we find

$$\overleftrightarrow{M}_c \overset{B}{\vec{V}}_c + \overleftrightarrow{C}_2 \vec{V}_c = \vec{F}_c \tag{20}$$

where

$$\begin{aligned} \overleftrightarrow{C}_2 &= \overleftrightarrow{C}_1 + \overleftrightarrow{M}_c \left[\vec{\Omega}\right] && (21) \\ &= \left[\vec{\Omega}\right] \overleftrightarrow{M}_c && (22) \end{aligned}$$

Expanding out (20) the explicit form of the equations of motion are

$$\begin{bmatrix} \overleftrightarrow{J}_c & \overleftrightarrow{0} \\ \overleftrightarrow{0} & m\overleftrightarrow{1} \end{bmatrix} \begin{bmatrix} \overset{B}{\vec{\omega}} \\ \overset{B}{\vec{v}}_c \end{bmatrix} + \begin{bmatrix} [\vec{\omega}]\overleftrightarrow{J}_c \vec{\omega} \\ m[\vec{\omega}]\vec{v}_c \end{bmatrix} = \begin{bmatrix} \vec{\tau}_c \\ \vec{f} \end{bmatrix} \tag{23}$$

Third Form of the Equations of Motion

In this section we derive the equations of motion of a single rigid body about an arbitrary point fixed on the body in terms of inertial rates-of-change.

For an arbitrary point, denoted o, fixed to a rigid body

$$\begin{aligned} \vec{v}_c &= \vec{v}_o + [\vec{\omega}]\vec{r}_{c/o} && (24) \\ \vec{\tau}_o &= \vec{\tau}_c + [\vec{r}_{c/o}]\vec{f} && (25) \end{aligned}$$

where $\vec{r}_{c/o}$ denotes the vector *from* point o *to* point c. Using spatial velocities and forces (24)-(25) can be expressed concisely as follows

$$\begin{aligned} \vec{V}_c &= \overleftrightarrow{T}\, \vec{V}_o && (26) \\ \vec{F}_o &= \overleftrightarrow{T}^T \vec{F}_c && (27) \end{aligned}$$

where

$$\overleftrightarrow{T} \triangleq \begin{bmatrix} \overleftrightarrow{1} & \overleftrightarrow{0} \\ -[\vec{r}_{c/o}] & \overleftrightarrow{1} \end{bmatrix} \tag{28}$$

and $[\vec{r}_{c/o}]^T = -[\vec{r}_{c/o}]$. Note that have used the fact that $\vec{\omega}_c = \vec{\omega}_o = \vec{\omega}$ and $\vec{f}_c = \vec{f}_o = \vec{f}$ in (26)-(27).

The absolute derivative of (26) is given by

$$\overset{N}{\vec{V}}_c = \overset{N}{\overleftrightarrow{T}}\, \vec{V}_0 + \overleftrightarrow{T} \overset{N}{\vec{V}}_o \tag{29}$$

Substituting (29) and (26) into (11) and pre-multiplying by $\overleftrightarrow{T}^T$ we find

$$\overleftrightarrow{M}_o \overset{N}{\vec{V}}_o + \overleftrightarrow{C}_3 \vec{V}_o = \vec{F}_o \tag{30}$$

where $\overleftrightarrow{M}_o = \overleftrightarrow{T}^T \overleftrightarrow{M}_c \overleftrightarrow{T}$, $\overleftrightarrow{C}_3 = \overleftrightarrow{T}^T \overleftrightarrow{M}_c \overset{N}{\overleftrightarrow{T}} + \overleftrightarrow{T}^T \overleftrightarrow{C}_1 \overleftrightarrow{T}$, and $\vec{F}_o = \overleftrightarrow{T}^T \vec{F}_c$. Expanding out the expression for $\overleftrightarrow{M}_o$ we find

$$\overleftrightarrow{M}_o = \begin{bmatrix} \overleftrightarrow{J}_c - [\vec{r}_{c/o}][\vec{r}_{c/o}] & m[\vec{r}_{c/o}] \\ -m[\vec{r}_{c/o}] & m\overleftrightarrow{1} \end{bmatrix} \tag{31}$$

To find an explicit formula for $\overleftrightarrow{C}_3$ note that

$$\overset{N}{\overleftrightarrow{T}} = \overset{B}{\overleftrightarrow{T}} + \left[\vec{\Omega}\right] \overleftrightarrow{T} - \overleftrightarrow{T} \left[\vec{\Omega}\right] \tag{32}$$

where $\overset{B}{\overleftrightarrow{T}} = \overleftrightarrow{0}$ since $\vec{r}_{c/o}$ is fixed in the body. Substituting (32) into the above expression for $\overleftrightarrow{C}_3$ and expanding yields

$$\overleftrightarrow{C}_3 = \begin{bmatrix} [\vec{\omega}]\overleftrightarrow{J}_c - \overleftrightarrow{J}_c[\vec{\omega}] - m[\vec{r}_{c/o}][\vec{\omega}][\vec{r}_{c/o}] + m[\vec{r}_{c/o}][\vec{r}_{c/o}][\vec{\omega}] & \overleftrightarrow{0} \\ -m[\vec{\omega}][\vec{r}_{c/o}] + m[\vec{r}_{c/o}][\vec{\omega}] & \overleftrightarrow{0} \end{bmatrix} \tag{33}$$

We immediately find

$$\overleftrightarrow{C}_3 \vec{V}_o = \begin{bmatrix} [\vec{\omega}]\overleftrightarrow{J}_c \vec{\omega} + m[\vec{r}_{c/o}][\vec{\omega}][\vec{\omega}]\vec{r}_{c/o} \\ m[\vec{\omega}][\vec{\omega}]\vec{r}_{c/o} \end{bmatrix} \tag{34}$$

The following fact will be used to simplify (34):

Proposition 1 *If $\vec{a}$ and $\vec{b}$ are arbitrary vectors then* $[\vec{a}][\vec{b}][\vec{b}]\vec{a} = -[\vec{b}][\vec{a}][\vec{a}]\vec{b}$.

Proof: For any vector $\vec{z}$, $[\vec{z}]\vec{z} = \vec{0}$. Letting $\vec{z} = \vec{a}\times\vec{b} = [\vec{a}]\vec{b}$ we find $[\vec{a}\times\vec{b}][\vec{a}]\vec{b} = \vec{0}$. Upon using the identity $[\vec{a}\times\vec{b}] = [\vec{a}][\vec{b}] - [\vec{b}][\vec{a}]$ and the fact that $[\vec{a}]\vec{b} = -[\vec{b}]\vec{a}$ the result follows. ∎

Applying Proposition 1 to (34), we find $m[\vec{r}_{c/o}][\vec{\omega}][\vec{\omega}]\vec{r}_{c/o} = -m[\vec{\omega}][\vec{r}_{c/o}][\vec{r}_{c/o}]\vec{\omega}$. As a result,

$$\overleftrightarrow{C}_3 \vec{V}_o = \begin{bmatrix} [\vec{\omega}](\overleftrightarrow{J}_c - m[\vec{r}_{c/o}][\vec{r}_{c/o}])\vec{\omega} \\ m[\vec{\omega}][\vec{\omega}]\vec{r}_{c/o} \end{bmatrix} \tag{35}$$

Note that $\overleftrightarrow{J}_o = \overleftrightarrow{J}_c - m\left[\vec{r}_{c/o}\right]\left[\vec{r}_{c/o}\right]$ by the parallel axis theorem.

Collecting together (31) and (35), the equations of motion of a rigid body about an arbitrary point fixed on the body in terms of inertial rates of change is

$$\begin{bmatrix} \overleftrightarrow{J}_o & m[\vec{r}_{c/o}] \\ -m[\vec{r}_{c/o}] & m\overleftrightarrow{1} \end{bmatrix} \begin{bmatrix} \overset{B}{\vec{\omega}} \\ \overset{B}{\vec{v}_o} \end{bmatrix} + \begin{bmatrix} [\vec{\omega}]\overleftrightarrow{J}_o\vec{\omega} \\ m[\vec{\omega}][\vec{\omega}]\vec{r}_{c/o} \end{bmatrix} = \begin{bmatrix} \vec{\tau}_o \\ \vec{f} \end{bmatrix} \tag{36}$$

Fourth Form of the Equations of Motion

In this section we derive two useful representations of the equations of motion of a rigid body about an arbitrary point fixed to the body in terms of body-fixed rates of change.

First Representation: Applying the transport formula $\overset{N}{\vec{V}_o} = \overset{B}{\vec{V}_o} + \left[\vec{\Omega}\right]\vec{V}_o$ to (30) we obtain

$$\overleftrightarrow{M}_o \overset{B}{\vec{V}_o} + \overleftrightarrow{C}_4 \vec{V}_o = \vec{F}_o \tag{37}$$

where

$$\overleftrightarrow{C}_4 = \overleftrightarrow{C}_3 + \overleftrightarrow{M}_o\left[\vec{\Omega}\right] \tag{38}$$

Substituting (33) into (38) and expanding we find

$$\overleftrightarrow{C}_4 = \begin{bmatrix} [\vec{\omega}]\overleftrightarrow{J}_c - m[\vec{r}_{c/o}][\vec{\omega}][\vec{r}_{c/o}] + m[\vec{r}_{c/o}][\vec{r}_{c/o}][\vec{\omega}] & m[\vec{r}_{c/o}][\vec{\omega}] \\ -m[\vec{\omega}][\vec{r}_{c/o}] & m[\vec{\omega}] \end{bmatrix} \tag{39}$$

It follows that

$$\overleftrightarrow{C}_4 \vec{V}_o = \begin{bmatrix} [\vec{\omega}]\overleftrightarrow{J}_c\vec{\omega} + m[\vec{r}_{c/o}][\vec{\omega}][\vec{\omega}]\vec{r}_{c/o} + m[\vec{r}_{c/o}][\vec{\omega}]\vec{v}_o \\ m[\vec{\omega}]\vec{v}_o + m[\vec{\omega}][\vec{\omega}]\vec{r}_{c/o} \end{bmatrix} \tag{40}$$

Applying Proposition 1 to the term $m[\vec{r}_{c/o}][\vec{\omega}][\vec{\omega}]\vec{r}_{c/o}$ in (40), we find that the equations of motion about an arbitrary point on a rigid body in terms of body-fixed rates-of-change are

$$\begin{bmatrix} \overleftrightarrow{J}_o & m[\vec{r}_{c/o}] \\ -m[\vec{r}_{c/o}] & m\overleftrightarrow{1} \end{bmatrix} \begin{bmatrix} \overset{B}{\vec{\omega}} \\ \overset{B}{\vec{v}_o} \end{bmatrix} + \begin{bmatrix} [\vec{\omega}]\overleftrightarrow{J}_o\vec{\omega} + m[\vec{r}_{c/o}][\vec{\omega}]\vec{v}_o \\ m[\vec{\omega}]\vec{v}_0 + m[\vec{\omega}][\vec{\omega}]\vec{r}_{c/o} \end{bmatrix} = \begin{bmatrix} \vec{\tau}_o \\ \vec{f} \end{bmatrix} \tag{41}$$

Note that (41) can be also be derived by applying the transport formula directly to (36).

Second Representation: An alternate representation of (41) results from applying the Jacobi identity $\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = \vec{0}$ to the term $m[\vec{r}_{c/o}][\vec{\omega}]\vec{v}_o$:

$$\begin{aligned} m[\vec{r}_{c/o}][\vec{\omega}]\vec{v}_o &= -[\vec{\omega}][\vec{v}_o]\vec{r}_{c/o} - [\vec{v}_o][\vec{r}_{c/o}]\vec{\omega} && (42) \\ &= [\vec{\omega}][\vec{r}_{c/o}]\vec{v}_o - [\vec{v}_o][\vec{r}_{c/o}]\vec{\omega} && (43) \end{aligned}$$

We immediately find

$$\begin{bmatrix} \overleftrightarrow{J}_o & m[\vec{r}_{c/o}] \\ -m[\vec{r}_{c/o}] & m\overleftrightarrow{1} \end{bmatrix} \begin{bmatrix} \overset{B}{\vec{\omega}} \\ \overset{B}{\vec{v}_o} \end{bmatrix} + \begin{bmatrix} [\vec{\omega}]\overleftrightarrow{J}_o\vec{\omega} + m[\vec{\omega}][\vec{r}_{c/o}]\vec{v}_o - m[\vec{v}_o][\vec{r}_{c/o}]\vec{\omega} \\ m[\vec{\omega}]\vec{v}_0 + m[\vec{\omega}][\vec{\omega}]\vec{r}_{c/o} \end{bmatrix} = \begin{bmatrix} \vec{\tau}_o \\ \vec{f} \end{bmatrix} \tag{44}$$

Adding the zero vector in the form of $m[\vec{v}_o]\vec{v}_o$ to (44) yields

$$\begin{bmatrix} \vec{J}_o & m[\vec{r}_{c/o}] \\ -m[\vec{r}_{c/o}] & m\overleftrightarrow{1} \end{bmatrix} \begin{bmatrix} \overset{B}{\vec{\omega}} \\ \overset{B}{\vec{v}_o} \end{bmatrix} +$$

$$\begin{bmatrix} [\vec{\omega}]\,\vec{J}_o\vec{\omega} + m\,[\vec{\omega}]\,[\vec{r}_{c/o}]\vec{v}_o - m\,[\vec{v}_o]\,[\vec{r}_{c/o}]\vec{\omega} + m[\vec{v}_o]\vec{v}_o \\ m\,[\vec{\omega}]\,\vec{v}_0 + m\,[\vec{\omega}]\,[\vec{\omega}]\,\vec{r}_{c/o} \end{bmatrix} = \begin{bmatrix} \vec{\tau}_o \\ \vec{f} \end{bmatrix}$$

After some manipulation the above equation can be expressed as follows

$$\overleftrightarrow{M}_o \overset{B}{\vec{V}}_o + \overleftrightarrow{C}_4\,\vec{V}_o = \vec{F}_o \tag{45}$$

where

$$\overleftrightarrow{C}_4 \;=\; -\left[\vec{\Omega}_V\right] \overleftrightarrow{M}_o \tag{46}$$

and

$$\left[\vec{\Omega}_V\right] \triangleq \begin{bmatrix} [\vec{\omega}] & [\vec{v}_o] \\ \overleftrightarrow{0} & [\vec{\omega}] \end{bmatrix} \tag{47}$$

Equation (45) is a coordinate-free version of the Lie group based equations of motion of a single rigid body described in [9].

Kinematic Equations

In order to provide a complete description of the motion of a single rigid body (i.e., a single spacecraft), a set of kinematic equations for each body is required. For a single unconstrained rigid body the kinematic differential equations (relative to the center-of-mass of the body) are given by $\overset{N}{\vec{r}_c} = \vec{v}_c$ and $\overset{N}{\overleftrightarrow{R}} = [\vec{\omega}]\,\overleftrightarrow{R}$. Here $\overleftrightarrow{R}$ denotes the rotation dyadic describing the orientation of $\mathcal{F}_B$ (with origin at the center-of-mass of the body c) relative to $\mathcal{F}_N$, $\vec{r}_c$ denotes the position of the center-of-mass of the rigid body relative to the origin of the inertial frame, and $\vec{v}_c$ denotes the absolute velocity of the center-of-mass of the rigid body relative to an inertially fixed observer. Note that a set of kinematic equations similar to those given above can be developed about an arbitrary point o fixed on the rigid body.

The Spatial Coriolis Dyadic

In the last section we developed the following four alternate forms of the equations of motion of a single rigid body:

$$\overleftrightarrow{M}_1 \overset{N}{\vec{V}}_c + \overleftrightarrow{C}_1\,\vec{V}_c \;=\; \vec{F}_c \tag{48}$$

$$\overleftrightarrow{M}_2 \overset{B}{\vec{V}}_c + \overleftrightarrow{C}_2\,\vec{V}_c \;=\; \vec{F}_c \tag{49}$$

$$\overleftrightarrow{M}_3 \overset{N}{\vec{V}}_o + \overleftrightarrow{C}_3\,\vec{V}_o \;=\; \vec{F}_o \tag{50}$$

$$\overleftrightarrow{M}_4 \overset{B}{\vec{V}}_o + \overleftrightarrow{C}_4\,\vec{V}_o \;=\; \vec{F}_o \tag{51}$$

where $\overleftrightarrow{M}_1 = \overleftrightarrow{M}_2 = \overleftrightarrow{M}_c$ and $\overleftrightarrow{M}_3 = \overleftrightarrow{M}_4 = \overleftrightarrow{M}_o$. Although the products $\overleftrightarrow{C}_1\,\vec{V}_c, \overleftrightarrow{C}_2\,\vec{V}_c, \overleftrightarrow{C}_3\,\vec{V}_o, \overleftrightarrow{C}_4\,\vec{V}_o$ are certainly unique, the *spatial Coriolis dyadic* $\overleftrightarrow{C}_i$ $i = 1,2,3,4$ is not. As will be shown, this is a consequence of the fact that $\overleftrightarrow{C}_i$ is itself a function of the spatial velocity.

For the purpose of dynamic modeling and simulation, the non-uniqueness of the spatial Coriolis dyadic is not an important issue as any admissible choice of $\overleftrightarrow{C}_i$ leads to the correct linear/angular accelerations. However, when developing nonlinear tracking control laws for separated spacecraft (see [2] for applications to underwater vehicles) the choice of the spatial Coriolis dyadic is critical. Specifically, $\overleftrightarrow{C}_i$ must be defined in such a way that it renders $\overset{N}{\overleftrightarrow{M}}_i - 2\,\overleftrightarrow{C}_i$ skew-symmetric. For example, in [12] a globally stable adaptive control law for robotic vehicles is designed that results in asymptotic tracking of a desired reference trajectory $q_d(t) \in \Re^n$ where q_d denotes specific generalized coordinates. The stability proof of the adaptive control law requires that $s^T(\dot{M} - 2C)s = 0$ where $s \in \Re^n$ is a function of both q and $\dot{q}$ and M is the system mass matrix [12]. As a result, the matrix representation of the Coriolis dyadic, denoted C, must be constructed in such a way to render the matrix $(\dot{M} - 2C)$ skew-symmetric[6]. The explicit relationship between the matrices M and C and the dyadics $\overleftrightarrow{M}$ and $\overleftrightarrow{C}$ for a multibody spacecraft is discussed in [9].

It is also known [9] that the equations of motion of a multibody spacecraft (i.e., a spacecraft consisting of a collection of hinge connected rigid bodies) inherit the skew-symmetry property from the equations of motion at the individual body level. As a result, it is important to define the appropriate spatial Coriolis dyadic $\overleftrightarrow{C}_i$ at the level of each individual rigid body. To this end, the following result is useful.

Proposition 2 *If* $\overleftrightarrow{C}_i$ *is skew-symmetric then* $\overset{N}{\overleftrightarrow{M}}_i - 2\,\overleftrightarrow{C}_i$ *is skew-symmetric.*

Proof: Recalling $\overset{N}{\overleftrightarrow{M}}_i = \left[\vec{\Omega}\right] \overleftrightarrow{M}_i - \overleftrightarrow{M}_i \left[\vec{\Omega}\right]$ it follows immediately that $\overset{N}{\overleftrightarrow{M}}_i = -\overset{N}{\overleftrightarrow{M}}{}_i^T$. Observing that the difference of two skew-symmetric tensors is skew-symmetric establishes the result. ■

[6]It is important to note that if $s = \dot{q}$, then $\dot{q}^T(\dot{M} - 2C)\dot{q} = 0$ irrespective of the skew-symmetry of $\dot{M} - 2C$. This statement is a property of finite-dimensional natural systems; see [9] for additional details and references. As a result, it is only in situations where $(\dot{M} - 2C)$ is pre- and post-multiplied by a column vector different from $\dot{q}$ (the typical case in control design) that C should be carefully defined.

We now discuss some specific choices of the spatial Coriolis dyadic that render $\overset{N}{\overleftrightarrow{M}_i} - 2\overleftrightarrow{C}_i$ skew-symmetric. Beginning with the fundamental form of the equations of motion (14) we find from expanding (16) that

$$\overleftrightarrow{C}_1 = \begin{bmatrix} [\vec{\omega}]\overleftrightarrow{J}_c - \overleftrightarrow{J}_c[\vec{\omega}] & \overleftrightarrow{0} \\ \overleftrightarrow{0} & \overleftrightarrow{0} \end{bmatrix} \tag{52}$$

Noting that $\overleftrightarrow{C}_1$ is skew-symmetric, it follows that the fundamental form of the equations of motion are manifestly skew-symmetric. We will denote $\overleftrightarrow{C}_1$ in (52) by $\overleftrightarrow{C}_1^{ss}$.

The Coriolis dyadic associated with the second form of the equations of motion (20) is not skew-symmetric. In order to develop a skew-symmetric representation we expand (22) and find

$$\overleftrightarrow{C}_2 = \begin{bmatrix} [\vec{\omega}]\overleftrightarrow{J}_c & \overleftrightarrow{0} \\ \overleftrightarrow{0} & m[\vec{\omega}] \end{bmatrix} \tag{53}$$

A skew-symmetric Coriolis dyadic results by modifying (53) as follows

$$\overleftrightarrow{C}_2^{ss} = \begin{bmatrix} [\vec{\omega}]\overleftrightarrow{J}_c - \overleftrightarrow{J}_c[\vec{\omega}] & \overleftrightarrow{0} \\ \overleftrightarrow{0} & m[\vec{\omega}] \end{bmatrix} \tag{54}$$

Note that although we have modified the Coriolis dyadic, the second form of the equations of motion (20) has not changed since $\overleftrightarrow{C}_2^{ss}\vec{V}_c = \overleftrightarrow{C}_2\vec{V}_c$. The above technique of adding the zero vector in a judicious fashion is the key to developing the appropriate spatial Coriolis dyadic for use in nonlinear spacecraft control.

The fourth form of the equations of motion leads to several admissible skew-symmetric forms. For example, although (39) is not skew-symmetric, a skew-symmetric representation of the fourth form of the equations of motion results from the following modification of (39)

$$\overleftrightarrow{C}_4^{ss} = \begin{bmatrix} [\vec{\omega}]\overleftrightarrow{J}_o - \overleftrightarrow{J}_o[\vec{\omega}] & m\left[\vec{r}_{c/o}\right][\vec{\omega}] \\ -m[\vec{\omega}]\left[\vec{r}_{c/o}\right] & m[\vec{\omega}] \end{bmatrix} \tag{55}$$

The skew-symmetric spatial Coriolis dyadic given in (55) can be used to construct another skew-symmetric form of the equations of motion. To this end consider the product

$$\overleftrightarrow{C}_4^{ss}\vec{V}_0 = \begin{bmatrix} [\vec{\omega}]\overleftrightarrow{J}_o\vec{\omega} - \overleftrightarrow{J}_o[\vec{\omega}]\vec{\omega} + m\left[\vec{r}_{c/o}\right][\vec{\omega}]\vec{v}_o \\ -m[\vec{\omega}]\left[\vec{r}_{c/o}\right]\vec{\omega} + m[\vec{\omega}]\vec{v}_o \end{bmatrix} \tag{56}$$

Applying the identity $[\vec{a}][\vec{b}]\vec{a} = [\vec{a}\times\vec{b}]\vec{a}$ to the term $[\vec{\omega}]\left[\vec{r}_{c/o}\right]\vec{\omega}$, using the identity $[\vec{a}]\vec{b} = -[\vec{b}]\vec{a}$, and subtracting the zero term $m[\vec{v}_o]\vec{v}_o$ yields

$$\overleftrightarrow{C}_4^{ss}\vec{V}_0 = \begin{bmatrix} -\left[\overleftrightarrow{J}_o\vec{\omega}\right]\vec{\omega} - \overleftrightarrow{J}_o[\vec{\omega}]\vec{\omega} + m\left[\vec{r}_{c/o}\right][\vec{\omega}]\vec{v}_o - m[\vec{v}_o]\vec{v}_o \\ -m\left[\vec{\omega}\times\vec{r}_{c/o}\right]\vec{\omega} - m[\vec{v}_o]\vec{\omega} \end{bmatrix} \tag{57}$$

To simplify (57) the following result is required:

Proposition 3 *If $\vec{a}, \vec{b}$ and $\vec{c}$ are arbitrary vectors then* $[\vec{a}][\vec{b}]\vec{c} = -[\vec{b}\times\vec{a}]\vec{c} + [\vec{c}\times\vec{a}]\vec{b}$

Proof: We find from rearranging the Jacobi identity $\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = \vec{0}$ that $[\vec{a}][\vec{b}]\vec{c} = -[\vec{b}][\vec{c}]\vec{a} - [\vec{c}][\vec{a}]\vec{b} = -[\vec{b}][\vec{c}]\vec{a} + [\vec{c}][\vec{b}]\vec{a} = -[\vec{b}](\vec{c}\times\vec{a}) + [\vec{c}](\vec{b}\times\vec{a}) = [\vec{c}\times\vec{a}]\vec{b} - [\vec{b}\times\vec{a}]\vec{c}$. ■

Applying Proposition 3 to the term $[\vec{r}_{c/o}][\vec{\omega}]\vec{v}_o$ in (57), we find after some rearranging

$$\overleftrightarrow{C}_4^{ss} = \begin{bmatrix} -[\overleftrightarrow{J}_o\vec{\omega}] + m\left[\vec{v}_o\times\vec{r}_{c/o}\right] & -m\left[\vec{\omega}\times\vec{r}_{c/o}\right] - m[\vec{v}_o] \\ -m\left[\vec{\omega}\times\vec{r}_{c/o}\right] - m[\vec{v}_o] & \overleftrightarrow{0} \end{bmatrix} \tag{58}$$

The form of the Coriolis dyadic given in (58) is similar to the result obtained by Fossen in [2].

3 Equations of Motion for Formations

In this section we demonstrate that the equations of motion of an entire formation of N rigid spacecraft has the same structure as the equations of motion of a single rigid spacecraft. In order to discuss collections of rigid bodies, the previous notation introduced for a single rigid body (cf. equation (48)) must be modified. To this end, the equations of motion of the i^{th} spacecraft $i = 1, 2, \ldots, N$ are denoted

$$\overleftrightarrow{M}_i(p)\overset{N}{\vec{V}_i}(p) + \overleftrightarrow{C}_i(p)\vec{V}_i(p) = \vec{F}_i(p) \tag{59}$$

where point p is either the center-of-mass c or a general point on the body o of the i^{th} spacecraft[7]. Here the Coriolis dyadic $\overleftrightarrow{C}_i(p)$ is assumed to be any admissible skew-symmetric dyadic (consistent with the point p) as discussed in the previous section.

By stacking the equations of motion of each spacecraft as follows:

$$\overleftrightarrow{\mathcal{M}} \triangleq Diag[\overleftrightarrow{M}_1(p), \overleftrightarrow{M}_2(p), \ldots, \overleftrightarrow{M}_N(p)] \tag{60}$$

$$\vec{\mathcal{V}} \triangleq col[\vec{V}_1(p), \vec{V}_2(p), \ldots, \vec{V}_N(p)] \tag{61}$$

$$\overleftrightarrow{\mathcal{C}} \triangleq Diag[\overleftrightarrow{C}_1(p), \overleftrightarrow{C}_2(p), \ldots, \overleftrightarrow{C}_N(p)] \tag{62}$$

$$\vec{\mathcal{F}} \triangleq col[\vec{F}_1(p), \vec{F}_2(p), \ldots, \vec{F}_N(p)] \tag{63}$$

we immediately find that the *global equations of motion of the formation* can be expressed as:

$$\overleftrightarrow{\mathcal{M}}\overset{N}{\vec{\mathcal{V}}} + \overleftrightarrow{\mathcal{C}}\vec{\mathcal{V}} = \vec{\mathcal{F}} \tag{64}$$

[7]Strictly $p = p_i$ because the equations of motion of each body can be expressed with respect to different reference points.

Note that equation (64), describing the global dynamics of the formation, has the same structure as the equations of motion of a single rigid body. Although we have used inertial derivatives in (64), body fixed derivatives can also be used. The following property of the formation equations of motion is of interest for the design of formation control laws:

Proposition 4 *If* $\overleftrightarrow{C}_i\,(p)$ *is skew-symmetric for* $i = 1, 2, \ldots, N$ *then* $\overset{N}{\overleftrightarrow{\mathcal{M}}} - 2\,\overleftrightarrow{\mathcal{C}}$ *is also skew-symmetric.*

Proof: The proof follows from applying Proposition 2 directly to the definitions of $\overleftrightarrow{\mathcal{M}}$ and $\overleftrightarrow{\mathcal{C}}$ given above. ■

As will be demonstrated in a later paper, the global (absolute) formation equations of motion (64) form the starting point for describing a formation as a virtual multi-body system with a branched-chain (or tree) topology [8]. For example, once a particular spacecraft has been designated as the leader (i.e., the basebody of the virtual multibody chain), the remaining spacecraft are analogous to the rigid links of a multibody chain. Each follower spacecraft is "attached" to the basebody spacecraft via a 6DOF free-free joint. Here, a free-free joint represents unconstrained motion between bodies; i.e., a free-free joint consists of a free 3 DOF prismatic joint and a free 3DOF spherical joint. As a result, in the virtual multi-body framework, the relative equations of motion of the formation become the primary concern. Due to space limitations, the relative equations of motion will be developed in a future paper.

4 Conclusions

In this paper we have developed a tensorial (i.e., coordinate-free) derivation of the equations of motion of a formation consisting of N spacecraft each modeled as a rigid body. The results presented here are the first step toward developing a coordinate-free architecture for formation flying dynamic modeling and control. Future work will address (1) characterizing the *relative* dynamics of formations using the concept of a virtual multi-body system, (2) developing the appropriate environmental disturbance models, (3) developing explicit techniques to linearize and perform sensitivity analysis on the formation equations of motion, and (4) developing the equations of motion for formations where each individual spacecraft is itself a multi-body system.

[8] A multibody system is called *branched-chain* or *tree topology* if it consists of a set of serial chains of bodies, each connected to a central basebody.

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References

[1] T.E. Carter, "State transition matrices for terminal rendezvous studies: brief survey and new example," *J. Guidance, Control, and Dynamics.* Vol. 21, No. 1, pp. 148-155, 1998.

[2] T.I. Fossen, *Guidance and Control of Ocean Vehicles.* Chichester: John Wiley, 1994.

[3] D.T. Greenwood, *Advanced Dynamics.* Cambridge: Cambridge University Press, 2003.

[4] P.C. Hughes, *Spacecraft Attitude Dynamics.* New York: John Wiley, 1986.

[5] W. Jaunzemis, *Continuum Mechanics.* New York: MacMillan, 1967.

[6] T.R. Kane, and D.A. Levinson, *Dynamics: Theory and Applications.* New York: McGraw-Hill, 1985.

[7] A. Kelkar, and S. Joshi, *Control of Nonlinear Multibody Flexible Space Structures: Lecture Notes in Control and Information Sciences, Volume 221.* New York, Springer, 1996

[8] H. Pan, and V. Kapila, "Adaptive nonlinear control for spacecraft formation flying with coupled translational and attitude dynamics," *Proc. IEEE Conf. on Decision and Control.* Orlando, FL, pp. 2057-2062, 2001.

[9] S.R. Ploen, "A skew-symmetric form of the recursive Newton-Euler algorithm for the control of multibody systems", *Proc. American Control Conf.* San Diego, pp. 3770-3773, 1999.

[10] J.E. Prussing, and B.A. Conway, *Orbital Mechanics.* New York, Oxford University Press, 1993.

[11] D. Scharf, F. Hadaegh, and B. Kang, "On the validity of the double integrator approximation in deep space formation flying," *Proc. Int. Symp. Formation Flying Missions and Technology.* Toulouse, France, 2002.

[12] J.J.E. Slotine, and W. Li, *Applied Nonlinear Control.* Englewood Cliffs, NJ: Prentice Hall, 1991.

[13] C. Truesdell, " Whence the law of moment of momentum?", *Essays in the History of Mechanics.* Springer-Verlag, New York, pp. 239-271, 1968.

[14] P.K.C. Wang, and F.Y. Hadaegh, "Coordination and control of multiple microspacecraft moving in formation," *J. of the Astronautical Sciences.* Vol. 44, No. 3, pp. 315-355, 1996.