Minimum Impulse Transfers to Rotate the Line of Apsides

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Introduction

Transfer between two coplanar orbits can be accomplished via a single impulse if the two orbits intersect. Optimization of a single-impulse transfer, however, is not possible since the transfer orbit is completely constrained by the initial and final orbits. On the other hand, two-impulse transfers are possible between any two terminal orbits. While optimal scenarios are not known for the general two-impulse case, there are various approximate solutions to many special cases. We consider the problem of an in-plane rotation of the line of apsides, leaving the size and shape of the orbit unaffected.

A Two-Impulse Approximation

A single impulse can be used to rotate the line of apsides of an orbit of eccentricity e and semi-major axis a through an angle $\Delta\omega$. Such a transfer requires a ΔV given by

$$\Delta V_{single} = 2e\sin\left(\frac{\Delta\omega}{2}\right)\sqrt{\frac{\mu}{a(1-e^2)}} \qquad (1)$$

where *e* is the eccentricity, *a* is the semi-major axis, and μ is the gravitational constant. In Figure 1, an initial and final orbit and their respective lines of apsides are shown. The impulse occurs either at the intersection of the orbits at apoapse or at the intersection at periapse, with ΔV given by Eq. (1).

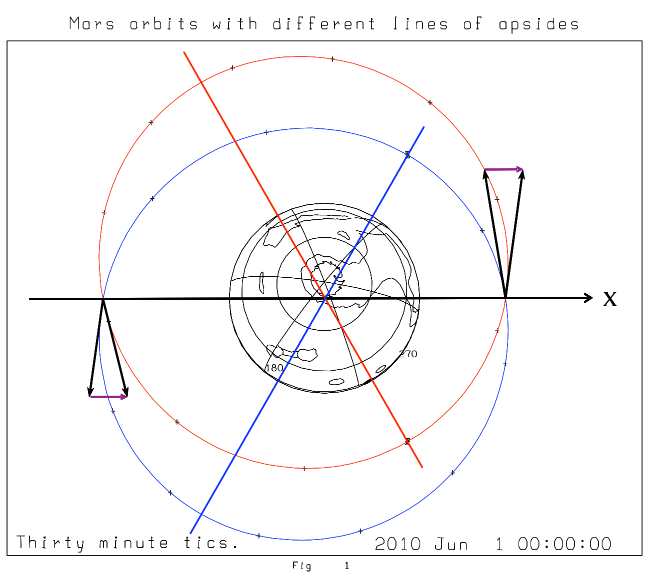

Figure 1: The two possible locations for a single-impulse rotation of apsides.

However, it has commonly been assumed that multiple impulses allow ΔV reduction. More specifically, a rule of thumb in common use is that a two-impulse transfer to rotate apsides uses half the total ΔV of a single impulse transfer for the same rotation. The basis of this rule of thumb seems to have come from Edelbaum's consideration of the problem of minimum impulse transfer for nearly circular orbits.[1] From his paper, linearization of the variation of parameter equations about a circular reference orbit gives the equations of motion, including

$$\frac{de_y}{du} = \sqrt{\frac{a}{\mu}}\left(2l_T \sin\theta - l_R \cos\theta\right) \tag{2}$$

where e_y is the y component of the eccentricity vector e, u is the time integral of thrust acceleration, θ is the angle of the maneuver point from the X-axis, and l_T and l_R are the circumferential and radial components of the direction vector of the maneuver respectively. Refer to Figure 2 for a graphical representation of Eq. (2).

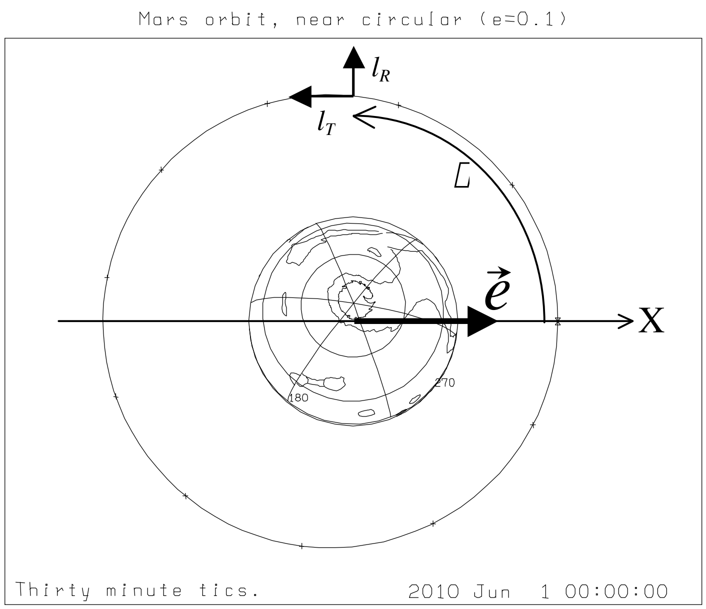

Figure 2: Eccentricity vector of a circular orbit.

From Eq. (2), it is clear that for nearly circular orbits a circumferential maneuver at θ = 90° is twice as effective as a radial maneuver at periapse. This simple observation has lead to our current rule of thumb that a two-impulse transfer requires half as much ΔV as a single-impulse transfer to rotate apsides:

$$\Delta V_{RoT} = \frac{\Delta V_{single}}{2} = e \sin\left(\frac{\Delta\omega}{2}\right)\sqrt{\frac{\mu}{a(1-e^2)}} \tag{3}$$

Lawden first considered the problem of optimal slewing of the orbital axis for elliptical orbits in 1962. In agreement with our rule of thumb, he states and assumes, without discussion, that a two-impulse transfer is more economical than a single impulse transfer.[2] He presents an algorithmic solution to the problem of optimizing the two-impulse transfer that requires the tedious, iterative satisfaction of six simultaneous equations. Furthermore, the equations give little insight as to the effect of eccentricity on the ΔV. It therefore remains of limited practical usefulness for mission design analysis. Our rule of thumb, while not optimal and only heuristically applicable to circular orbits, is at least an explicit solution. A better understanding of the behavior of the rule at higher eccentricities will allow an improvement of the approximation and greater applicability to include elliptical orbits.

Analysis

Our assumption is that a good approximation of the optimal ΔV needed to rotate an orbit through an angle Δω along the line of apsides is given by half of Eq. (1). The accuracy of this approximation, as well as the effects of semi-major axis and eccentricity, was examined via numeric comparisons with results from the trajectory optimization program CONSAT. CONSAT, developed at JPL by Carl Sauer, uses patched conic analysis and finite parameter optimization to optimize interplanetary trajectories. A slightly inclined (i = 10°), near circular orbit (a = 5000, e = 0.05) of Mars (also Ω = 0°) was used as a baseline. Allowing CONSAT to search on the positions of the true anomaly for the initial and final orbits, the initial baseline orbit was rotated around the orbit normal by 10°. This line-of-apsides rotation was then extended up to Δω = 340° for a = 5000 km and then for a = 7000 km. These constraints were then replicated for 5 other increasingly larger eccentricities. Table 1 presents a summary of the cases considered and the results of the comparison. Results for e = 0.05, as well as some other very near circular cases, are not shown.

Summary of Results for Apsides Rotation at Mars (Gm = 42828 km3/s2)
(Inclination = 10 deg, longitude of ascending node = 0 deg)
(Optimal transfer ΔV values were calculated using CONSAT Ver. 3.14, and estimated ΔV values were calculated from the approximation ΔV=e*sin(Δw/2)*SQRT(Gm/a*(1-e^2)).)

Δω (degrees)	a = 7400 e = 0.15	a = 5000 e = 0.15	a = 7400 e = 0.2	a = 5000 e = 0.2	a = 7400 e = 0.4	a = 5000 e = 0.4	a = 7400 e = 0.6	a = 5000 e = 0.6	a = 7400 e = 0.8	a = 5000 e = 0.8
10	0.993	0.993		0.989	0.961	0.961	0.908	0.908	0.794	0.794
20	0.990	0.990	0.984	0.984	0.952	0.952	0.893	0.893	0.771	0.771
40	0.983	0.983	0.976	0.976	0.935	0.935	0.865	0.865	0.729	0.729
60	0.978	0.978	0.968	0.968	0.919	0.919	0.840	0.840	0.696	0.696
80	0.972	0.972	0.961	0.961	0.905	0.905	0.820	0.820	0.670	0.670
100	0.968	0.968	0.955	0.955	0.894	0.894	0.803	0.803	0.650	0.650
120	0.964	0.964	0.951	0.951	0.885	0.885	0.791	0.791	0.635	0.635
140	0.962	0.962	0.947	0.947	0.878	0.878	0.782	0.782	0.626	0.626
160	0.960	0.960	0.945	0.945	0.874	0.874	0.777	0.777	0.620	0.620
180	0.959	0.959	0.944	0.944	0.873	0.873	0.775	0.775	0.618	0.618
200	0.960	0.960	0.945	0.945	0.874	0.874	0.777	0.777	0.620	0.620
220	0.962	0.962	0.947	0.947	0.878	0.878	0.782	0.782	0.626	0.626
240	0.964	0.964		0.951	0.885	0.885	0.791	0.791	0.635	0.635
260	0.968	0.968		0.955	0.894	0.894	0.803	0.803	0.650	0.650
280	0.972	0.972	0.961	0.961	0.905	0.905	0.820	0.820	0.670	0.670
300	0.978	0.978			0.919	0.919	0.840	0.840	0.696	0.696
320	0.983	0.983		0.976	0.935	0.935	0.865	0.865	0.729	0.729
340	0.990	0.990			0.952	0.952	0.893	0.893	0.771	0.771

Notes:
All blank cells in the tables are instances when CONSAT did not converge.

Table 1: Ratio of Optimal Transfer ΔV to Estimated ΔV

In examining Table 1, it is clear that our rule of thumb provides a consistently conservative approximation to the most optimal ΔV. The ratios are identical for each value of a at every value of e indicating that the approximation is independent of semi-major axis. It was expected from the heuristic argument derived from Edelbaum that the approximation would be very good for nearly circular orbits. It is quite clear, however, that the approximation becomes less accurate with increasing eccentricity. For example, the ratio of optimal transfer ΔV to estimated ΔV for a 10° rotation of apsides at a = 7400 and e = 0.15 is 0.993, but as eccentricity increases to e = 0.8 the ratio decreases to 0.794 for the same amount of rotation. It is also clear that across all eccentricities the error increases with the amount of rotation and is maximal at the maximum rotation, θ = 180°.

The analysis also shows that in addition to the semi-major axis the direction of the apsides rotation does not affect the ΔV needed. In Table 1 it can be seen that for each case the ratios are symmetric around Δω = 180°. In the most optimal cases the same transfer orbit is essentially used for rotating in either direction. CONSAT outputs showed that only the order and direction of the maneuvers need to be reversed in order to rotate in the opposite direction. In Figure 3, for example, a 120° rotation in the counterclockwise direction can be accomplished by the ΔV1 and ΔV2 impulses along the solid red transfer orbit as shown. However, to rotate 120° in the opposite direction a –ΔV2 maneuver to get onto the dashed green transfer orbit is needed before a –ΔV1 maneuver to get off the transfer orbit and into the final rotated ellipse.

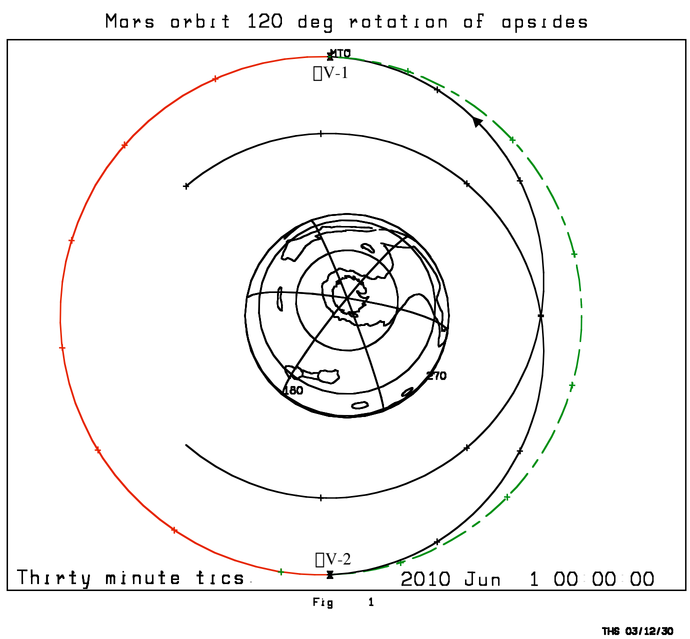

Figure 3: 120° rotation of apsides in either direction.

Bounding the Error

The percentage error for the rule of thumb is dependent on the eccentricity and the amount of rotation in the line of apsides.

Although we do not know an analytic formula for the general two-impulse case, there is an analytic formula for an optimal 180° rotation. Consider the scenario where the transfer orbit begins with circularization at the apoapse of the ellipse and 180° later restores the peripase altitude. The ΔV required is given by

$$\Delta V_{180} = 2\left(1 - \sqrt{1 - e}\right)\sqrt{\frac{\mu}{a(1+e)}} \qquad (4)$$

Lawden found for intersecting initial and final orbits that have aligned axes, the optimal transfer orbit will be tangential to both the initial and final orbits at an apse on each and will pass through the farther periapse (which is actually the apoapse in our case).[2] Thus the transfer scenario assumed by Eq. (4) is exactly the optimal 180° rotation scenario.

The ratio of Eq. (4) to the rule of thumb for $\Delta\omega$ = 180° given in Eq. (3) is

$$\frac{\Delta V_{180}}{\Delta V_{RoT}} = \frac{\sqrt{1-e}+\sqrt{1+e}}{1+\sqrt{1+e}} \tag{54}$$

The ratio given by Eq. (5) agrees with the values listed in Table 1 for a 180° rotation up to five decimal places. A quadratic fit using this ratio and a value of 1.000 for no rotation ($\Delta\omega$ = 0° and $\Delta\omega$ = 360°) gives an excellent fit through the data points at low eccentricities. At higher eccentricities, on the other hand, the fit was only tight around $\Delta\omega$ = 180°. To gain a better fit across all eccentricities the restriction that the endpoints have a value of 1.000 was removed. A polynomial least squares fit through the data was then used to determine initial ratios at $\Delta\omega$ = 0°. It was then observed that $(1 - \text{ratio}_{\Delta\omega=0})/(1 - \text{ratio}_{\Delta\omega=180}) \approx 0.5e$. A new quadratic fit, with the endpoints not artificially set to 1.000 but instead including this additional effect of eccentricity, showed a more consistent fit. Figures 4 and 5 show the accuracy of the fits. The marked points are data from Table 1 and the unmarked curves are quadratic fits through the center and endpoints.

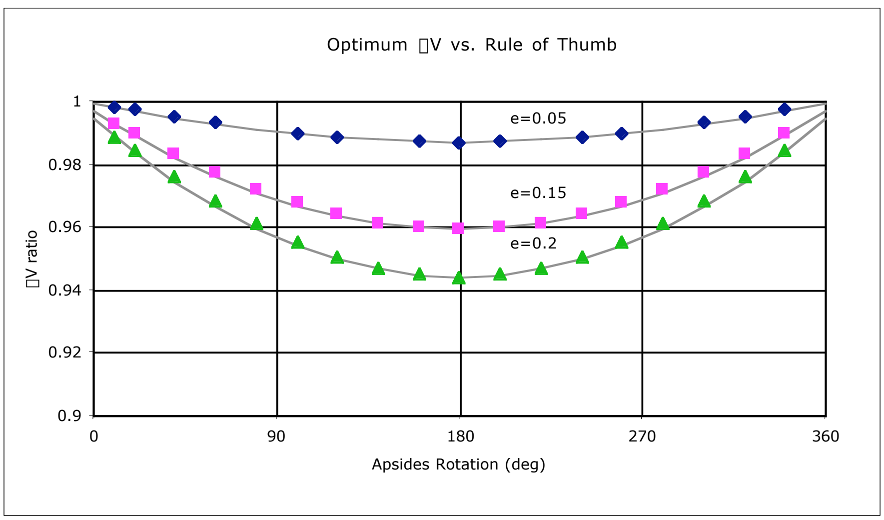

Figure 4: Optimum ΔV vs. an Improved Rule of Thumb of Baseline Orbit at e = 0.05, e = 0.15 and e = 0.2

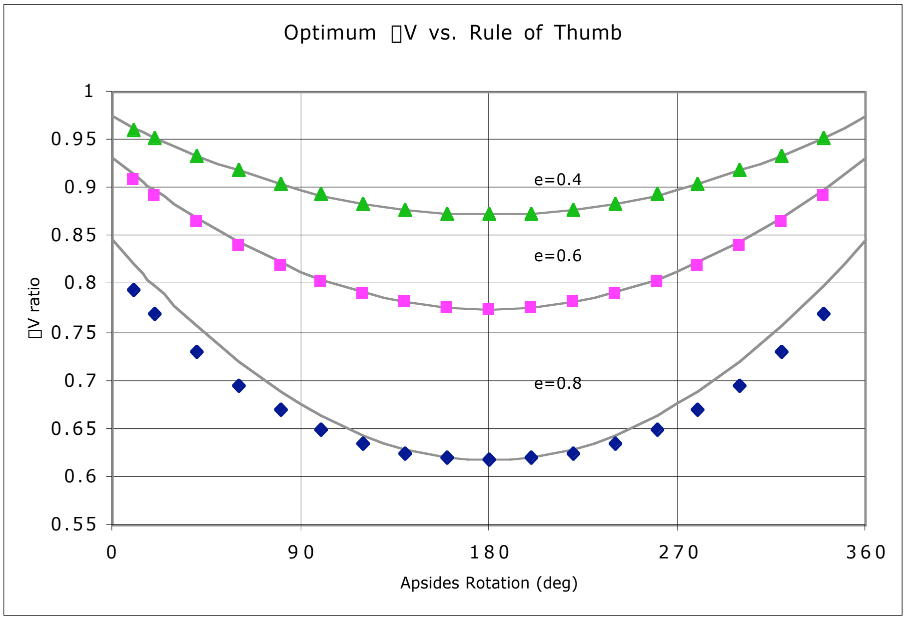

Figure 5: Optimum ΔV vs. an Improved Rule of Thumb of Baseline Orbit at e = 0.4, e = 0.6 and e = 0.8

Therefore, an improved ΔV estimate would be given by

$$\Delta V = \left[\left(\frac{\Delta\theta - 180}{180} \right)^2 (1 - 0.5e) + \frac{\sqrt{1-e} + \sqrt{1-e}}{1 + \sqrt{1-e}} \left[1 - \left(\frac{\Delta\theta - 180}{180} \right)^2 + 0.5e \left(\frac{\Delta\theta - 180}{180} \right)^2 \right] \right] \cdot \Delta V_{RoT} \quad (6)$$

Using the estimation rule of Eq. (5), the ΔV cost of rotating the line of apsides of a 6000 km x 4000 km orbit of Mars was also determined. The plot of the ΔV costs is shown in Figure 6. It is high by 1 m/s at the low end but is otherwise spot on over the rest of the curve when compared with the true optimal ΔV found by CONSAT.

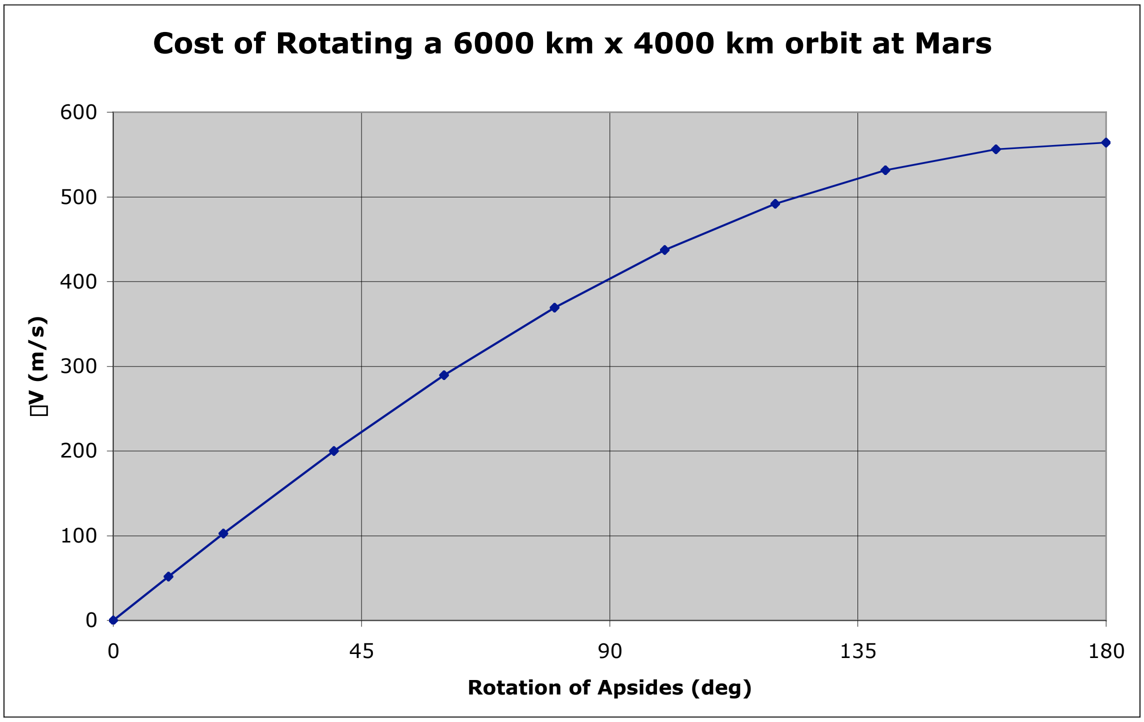

Figure 6: Using our new ΔV formula to estimate the cost of rotating an orbit at Mars. This is less than half a meter per second high (i.e., conservative) for the five lowest non-zero points on the left and much closer than that for the other points.

Conclusion

We have documented here an old and rather arcane bit of astrodynamics folklore, which gives an easy rule of thumb for the ΔV needed to rotate an orbit's line of apsides in the plane of the orbit. By comparing this rule of thumb to actual optima we have found an improved estimation formula, which can be used when more precision is needed in the design of space missions.

Acknowledgments

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