

Fluctuations in a ^4He Josephson Junction Array near the Lambda Point

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We present a formal thermodynamic treatment of the Josephson effect. We show that the current I_s and the phase difference ϕ are thermodynamic conjugate variables. Quantitative expressions for the fluctuations in I_s and ϕ , as well as their power spectral densities, are derived. With these, we explain why the Josephson effect is observable in a ^4He near the lambda transition despite the presence of strong fluctuations there. Treatment of thermodynamic stability and the thermal activation to the region of instability are also presented.

PACS numbers: 05.40.+j, 05.70.Ln, 67.57.Bc, 74.50.+r

One of most astounding accomplishment of modern physics is the establishment that all the condensates in a Bose liquid can be represented by a single wave function $\psi = \eta e^{i\Phi}$ with only two parameters – an amplitude η , and a quantum phase Φ . In superfluid ^4He , η is related to the superfluid density by $\rho_s = \eta^2 m$, and the superfluid velocity is related to $\nabla\Phi$ by: $u_s = (\hbar/m)\nabla\Phi$, where m is the mass of a ^4He atom. With this, many interesting quantum phenomena such as flux quantization, quantized vortices and the Josephson effect can be explained.

Although a lot is known about quantum fluid at very low temperature T where the effects of thermodynamic fluctuations are small, there are still unresolved questions near the lambda transition temperature T_λ . For example it was suspected that the Josephson effect in ^4He would be washed out by fluctuations near T_λ . Its recent observation [1] was therefore quite surprising and motivated us to take a closer look at the fluctuations in the junction. We are able to derive quantitative expressions for the fluctuations in I_s and ϕ . With these expressions, the existence of the effect is explained.

Zimmermann [3] gave an argument of why the Josephson effect might not be observable near T_λ . Consider a junction consisting of a single orifice of size $\ell \times \ell \times \ell$. The junction obeys the Josephson relation: $i_s = i_c \sin \phi$, where i_c is the critical current. The Josephson coupling energy is given by: $E_J^1(\phi) = (\hbar i_c / m)(1 - \cos \phi)$. It is generally accepted that [2] the condition for existence of the Josephson effect is $\|E_J^1\| > k_B T$, where $\|E_J^1\| = \hbar i_c / m$ is the amplitude of $E_J^1(\phi)$, k_B is Boltzmann's constant. We evaluate the quantity: $\|E_J^1\| / (k_B T) = \hbar i_c / (m k_B T)$ by noticing that $i_c \approx \rho_s u_s \ell^2$, where ρ_s is the

superfluid density in the junction, which is different from its bulk value - ρ_s^{Bulk} , due to finite size effect; u_s is the superfluid velocity in the junction when $\phi = \pi/2$ and therefore $u_s = \pi\hbar/(2\ell m)$. In ^4He , there is also the relation that: $\xi\rho_s^{Bulk} = m^2k_B T/\hbar^2$, where ξ is the correlation length. Thus:

$$\frac{\|E_J^1\|}{k_B T} = \left(\frac{\pi\ell}{2\xi}\right) \cdot \left(\frac{\rho_s}{\rho_s^{Bulk}}\right) \quad (1)$$

One condition required for the DC Josephson effect is $\ell < \xi$, and when this is met, $\rho_s < \rho_s^{Bulk}$ due to finite size effect. Therefore the two conditions $\ell < \xi$ and $\|E_J^1\| > k_B T$ cannot be simultaneously met, both of which are required.

One simple way to reconcile this is the observation that many orifices were used in the experiment. Furthermore, each orifice is a slit of area $\ell_1 \times \ell$ rather than a square of size $\ell \times \ell$. This greatly increased the total critical current and the Josephson coupling energy due to the increase in the cross section area. Let N_{slit} be the number of slits. We define an effective number $N = N_{slit}\ell_1/\ell$ to account for the increase in the cross section area. Thus N can be thought of as the effective number of $\ell \times \ell \times \ell$ orifices in parallel. Here we neglect the small difference in confinement geometry between square holes and slits. The Josephson energy of the array is $E_J = NE_J^1$ and $\frac{\|E_J\|}{k_B T} = N\left(\frac{\pi\ell}{2\xi}\right) \cdot \left(\frac{\rho_s}{\rho_s^{Bulk}}\right)$. For large N , both $\ell < \xi$ and $\|E_J\| > k_B T$ can be satisfied, and Josephson effect is possible. This explanation is consistent with the results of ref. [1], where the measurement shows

that $\|E_J\|/k_B T$ are 220 and 39 for $T - T_\lambda$ of 61 and 28 μK respectively. While this is a plausible explanation, it does not explain why all the orifices should function together and behave as a single junction with an enhanced critical current $I_c = Ni_c$. It gives no hint as to what would happen when one gets even closer to T_λ , so that even with the large N , $\|E_J\|/k_B T < 1$. To gain further insight, we examine the thermodynamics of the system.

The thermodynamics treatment begins with the realization that there is a different way to do work in superfluid. It is possible to move a porous material with the normal fluid locked inside. Therefore work is done only to the normal component while the superfluid passes through unaffected. One can write: $d(\text{Work}) = (\text{Force})dz = (dJ_n/dt)dz = (dz/dt)dJ_n = u_n dJ_n$, where z is the distance moved, $J_n = \rho_n u_n V$ is the normal fluid momentum, u_n is the normal fluid velocity and V is the volume of the sample. Thus the first law of thermal dynamics in the superfluid frame is [4]: $dE^s = TdS + u_n dJ_n$. Most experiments are performed in a frame where the normal component is at rest. Using Galilean transformation $E^n = E^s + V\rho(u_n - u_s)^2/2 - J_n(u_n - u_s)$, the first law in the normal fluid frame is: $dE^n = TdS + J_s du_s$, where $J_s = V\rho_s u_s$ is the superfluid momentum. For an orifice with cross section area A and thickness ℓ , it is more convenient to use $I_s = \rho_s u_s A$ and $\phi = (m/\hbar)u_s \ell$ as the conjugate variables. The free energy, $dF(T, u_s) = -SdT + J_s du_s$, becomes: $dF(T, \phi) = -SdT + (\hbar I_s/m)d\phi$, where I_s/m is also the particle current, which in superconductor can be written as $I_s''/2e$, where I_s'' is the electrical supercurrent. Thus most of the calculations here can be adapted to superconductor by replacing I_s/m with $I_s''/2e$.

It is possible to write down the fluctuations of ϕ and the conditions of stability by analogy to the gaseous system. In the case of a gaseous system, the work term is: $d(\text{Work}) = -PdV$, where P is the pressure, V is the volume. When P is kept constant the Gibbs free energy $G_1(T, P)$ can be expanded around its equilibrium value G_o as [5], $G_1 = G_o + (\partial G_1 / \partial V)_T \Delta V + (1/2)(\partial^2 G_1 / \partial V^2)_T (\Delta V)^2 + \dots$. The condition for stability requires G_1 to have a minimum at G_o . Giving the condition $(\partial^2 G_1 / \partial V^2)_T = -(\partial P / \partial V)_T > 0$. The probability for finding the system with a volume between V and $V + dV$ is proportional to $e^{-G_1 - G_o / k_B T} dV$, leading to the volume fluctuations: $\langle \delta V^2 \rangle = -k_B T (dV / dP)_T$. For superfluid flow in an orifice, the work term is: $d(\text{Work}) = (\hbar I_s / m) d\phi$, thus one can make the substitutions $-(\hbar I_s / m) \rightarrow P$ and $\phi \rightarrow V$. When I_s is kept constant, the stability condition and the phase fluctuations are:

$$\partial I_s / \partial \phi > 0, \quad (2)$$

$$\langle \delta \phi^2 \rangle = k_B T (m / \hbar) \cdot (dI_s / d\phi)^{-1}. \quad (3)$$

When the ϕ is kept constant, the fluctuations in I_s are $\langle \delta I_s^2 \rangle = k_B T (m / \hbar) \cdot (dI_s / d\phi)$.

Let us first consider the region far below T_λ where $\xi < \ell$. In this region, the work term can be integrated to give $\text{Work} = \rho_s u_s^2 V / 2$, and therefore is just the kinetic energy of superfluid flow. It is useful to define a hydrodynamic inductance L in such a way that $\text{Work} = LI_s^2 / 2$. Thus $L = \ell / (A \rho_s)$. Because $L \propto \ell / A$, and the electrical resistance of a resistor has the same geometric dependence, when orifices are connected in parallel or in

series, the resulting inductance is calculated the same way as electrical resistors. The large open geometry in both sides of the orifices in ref. (1) ensures that the values of inductance of the fluid connecting the orifices together are small compared to the inductance of the orifices so that the orifices are indeed connected in parallel.

In the region of $\xi < \ell$, the I_s is proportional to ϕ . The stability condition is satisfied; $\langle \delta I_s^2 \rangle = (\hbar/m)^2 \cdot (\xi A/\ell) \cdot (\rho_s \rho_s^{Bulk})$, and

$$\langle \delta \phi^2 \rangle = (\xi \ell / A) \cdot (\rho_s^{Bulk} / \rho_s) = \xi L \rho_s^{Bulk} \quad (4)$$

Equation (4) above establishes that $\langle \delta \phi^2 \rangle$ scales with L the same way Johnson voltage noise scales with resistance, i.e. $\langle \delta(Voltage)^2 \rangle \propto (R)$. The computation of the phase fluctuations for a network of hydrodynamic inductors is analogous to the computation of the Johnson noise from a network of resistors. The fluctuations in phase in each side of the orifices are small (due to the small inductance) compared with the fluctuations in the phase difference across the junction. Therefore the phase on each side of the orifices must be locked together at any instance in time. This is a formal derivation of the assertion that a unique phase exists everywhere inside a bulk superfluid. With $\xi > 10^{-8} \text{ cm}$, $\delta \phi_{rms} / \phi > 10^{-4}$ for centimeter-size system. Such a large phase fluctuations is quite surprising considering that for most other system the fractional fluctuations are approximately $1/\sqrt{N_{Particles}} \approx 10^{-11}$.

We next turn to the more interesting case close to T_λ , where $\ell < \xi$. In this region, $I_s = I_c \sin \phi$, where $I_c = Ni_c$ is the effective critical current through the array of

orifices. The stability condition given in Eq. (2) implies that $I_s(\phi)$ is stable only in the region where $\partial I_s/\partial\phi > 0$, i.e. $0 < \phi < \pi/2$ and $3\pi/2 < \phi < 2\pi$. In the range $\pi/2 < \phi < 3\pi/2$, ϕ is not stable against small fluctuations due to the violation of Le Chaterler's principle. It is possible for I_s and ϕ to traverse this region, but not dwell in it in steady state. The fluctuations in I_s and ϕ discussed in the following are only defined in the stable region. Using Eq. (3), when $I_s = \text{const.}$,

$$\langle \delta\phi^2 \rangle = \frac{k_B T m}{\hbar I_c \cos \phi}, \quad (5)$$

and when $\phi = \text{const.}$, $\langle \delta I_s^2 \rangle = k_B T m I_c \cos \phi / \hbar$.

For a single $\ell \times \ell \times \ell$ orifice, $\langle \delta\phi_1^2 \rangle = k_B T m / (\hbar i_c \cos \phi)$. After N such junctions are connected in parallel, the fluctuations should be reduced by a factor of \sqrt{N} as long as the spacing between the orifices is larger than ξ , so that their fluctuations are uncorrelated. Because the phases in each side of the orifices are locked together, it forces the phase differences across all the orifices to be averaged together at any instance. Thus when $I_c = N i_c$, Eq. (5) implies that $\langle \delta\phi^2 \rangle$ is reduced by $1/N$. The array behaves as one single junction with $I_c = N i_c$ and $E_J = N E_J^1$. In the following, we show that the reduction in phase fluctuations is responsible for making the Josephson effect observable.

Integrating the work term $d(\text{Work}) = (\hbar/m) I_s d\phi$ from 0 to ϕ gives: $E_J(\phi) = (\hbar I_c / m)(1 - \cos \phi)$. From Eq. (5) the condition for existence of the Josephson

effect: $\|E_J\| > k_B T$ is the same as $\langle \delta\phi^2 \rangle < 1$ at $\phi = 0$, suggesting that as $\delta\phi_{rms}$ becomes large, ϕ may fluctuate around a circle. When this occurs a large fraction of the time, ϕ averages to zero. Josephson effect would not be observable.

In the above analysis ($\delta\phi^2$) is assumed to be small enough that it is proportional to the slope of a response function ($d\phi/dI_s$). But as $\langle \delta\phi^2 \rangle \rightarrow 1$, this assumption is not valid. To develop the thermodynamics of large fluctuations, we notice that at constant current, the proper free energy to use is: $G(T, I_s) = F(T, \phi) - \hbar I_s \phi / m$, giving:

$$G(T, I_s) = F(T, 0) + (\hbar I_c / m)(1 - \cos \phi) - \hbar I_s \phi / m \quad (6)$$

Thus when I_s is held constant, the excess in the free energy when the phase deviates from its equilibrium value of $\phi_e = \sin^{-1}(I_s / I_c)$ by an amount of $\delta\phi$ is:

$$\begin{aligned} \Delta G(T, I_s) &= (\hbar I_c / m)[1 - \cos(\phi_e + \delta\phi)] - \hbar I_s (\phi_e + \delta\phi) / m - (\hbar I_c / m)(1 - \cos \phi_e) + \hbar I_s \phi_e / m \\ &= \Delta G(\phi_e, \delta\phi) = \|E_J\| [\cos \phi_e \cdot (1 - \cos \delta\phi) + \sin \phi_e \cdot (\sin \delta\phi - \delta\phi)] \end{aligned} \quad (7)$$

Therefore the probability of observing a deviation of $\delta\phi$ is: $P_r(\phi_e, \delta\phi) \propto e^{-\Delta G / k_B T}$, which for large $\|E_J\| / k_B T$ and small ϕ_e , gives a Gaussian function with a width given by Eq. (5) above. For $\phi_e > 0$, $P_r(\phi_e, \delta\phi)$ has a maximum at $\delta\phi = 0$, and decreases monotonically in the negative $\delta\phi$ direction. In the positive $\delta\phi$ direction, it decreases to a minimum and then diverges for large $\delta\phi$. Due to symmetry, only $\phi_e > 0$ is discussed

without loss of generality. This divergence is not of concern, because large $\delta\phi$ brings the phase into the unstable region where fluctuations are not well defined. But to continue the discussion, one must establish the region of validity for $P_r(\phi_e, \delta\phi)$. This is best done by analogy with the case of the rigid pendulum [6] shown in the inset of Figure 1. Here the constant applied I_s in the Josephson junction is analogous to a constant applied torque. If a constant torque is applied to balance the pendulum at an equilibrium angle θ_e , it is stable against small perturbation for $0 < \theta_e < \pi/2$, but unstable for $\pi/2 < \theta_e < 3\pi/4$. This is an analogy to the stability conditions discussed above. As long as the pendulum is released at an angle $\theta < \theta_c$, where $\theta_c = \pi - \theta_e$ (or an angle of $\pi - 2\theta_e$ away from θ_e in the positive direction), it will converge back to its original position. If released at a larger angle, it will roll around the circle. Therefore if fluctuations occasionally bring the pendulum to a position larger than θ_c , it will occasionally roll around. The behavior thus becomes intermittent. The intermittency increases with fluctuations until the motion becomes chaotic. Thus the condition for well-behaved fluctuations is $\delta\phi < \pi - 2\phi_e$.

Because $P_r(\phi_e, \delta\phi) \rightarrow 0$ for negative $\delta\phi$, the exact choice of the minimum $\delta\phi$ is not important as long as the probability density is small there. We choose $-\pi < \delta\phi$ for convenience. With the range of validity set at $-\pi < \delta\phi < \pi - 2\phi_e$, $P_r(\phi_e, \delta\phi)$ can be computed as: $P_r(\phi_e, \delta\phi) = e^{-\Delta G(\phi_e, \delta\phi)/k_B T} / D(\phi_e)$, where $D(\phi_e)$ is the normalization constant given by $D(\phi_e) = \int_{-\pi}^{\pi - 2\phi_e} e^{-\Delta G/k_B T} d\delta\phi$. For small $P_r(\phi_e, \pi - 2\phi_e)$, the above normalization introduces small error. But $P_r(\phi_e, \pi - 2\phi_e)$ determines the probability of thermal activation into the unstable region. Therefore an experimenter can choose a

criteria of how long to wait before declaring the system stable. This is equivalent to choosing a critical probability density P_c . Thus the condition $P_r(\phi_{\max}, \pi - 2\phi_{\max}) = P_c$ defines a maximum phase ϕ_{\max} and a maximum current $I_{\max} = I_c \sin \phi_{\max}$ that can be maintained across the junction under phase fluctuations. As T is raised and fluctuations increases, ϕ_{\max} and I_{\max} decreases, until they reach values of zeroes at $T = T_c$. Above T_c but below T_λ , although I_s and ϕ undergo chaotic motions, our analysis implies that quantum mechanical order is maintained, and the fluid in the orifice remains superfluid.

Figure 1 shows a plot of $P_r(\phi_e, \delta\phi)$ for several values of ϕ_e and $\|E_J\|/k_B T$. We chose $P_c = 0.1$ /radian, and computed $\|E_J\|/k_B T$ as a function of ϕ_{\max} . The results are shown in Figure 2. From this plot, $\phi_{\max} \rightarrow 0$ when $\|E_J\|/k_B T \rightarrow 0.425$, verifying the general assertion that as $\|E_J\|/k_B T \rightarrow 1$, Josephson effect becomes unobservable.

For small fluctuations the fluctuation dissipation theorem [7] gives the power spectral density (PSD) of z as $z_{PSD}(f) = 2k_B T \text{Im} \left[z(f)/F^{ext}(f) \right] / (\pi f)$, where f is the frequency, $F^{ext}(f)$ is a small amplitude AC drive and $z(f)$ is the AC response. An example of its application is the spring-mass system with a viscous damper shown in the inset of Figure 2. The solution z_{PSD} is used in the design of vibration isolation system for the detection of gravity wave [8]. This solution can be applied to the case of the Josephson junction, since the equation of motion of a rigid pendulum [6], for small amplitude oscillation, can be mapped to that of a the spring-mass. Here the small AC driving torque $\tilde{\Gamma}$ about a constant torque Γ_e for a rigid pendulum is analogous to the

small AC current \tilde{I} about a DC term I_e of a Josephson junction, or the AC driving force F^{ext} of a spring-mass oscillator. The following is the generalized equation of motion.

$$\frac{d^2 z_n}{dt^2} + \gamma_n \frac{dz_n}{dt} + \omega_{on}^2 z_n = \frac{F_n^{ext}}{M_n}, \quad (8)$$

where the subscripts $n=1,2,3,4$ denotes the spring-mass, the rigid pendulum, the superconductor and the superfluid systems respectively. The first, second and third terms originate respectively from the inertial force, the frictional force and the force F that does work to the system. The external applied force F^{ext} supplies the balance that maintains a steady state oscillation. At resonance, F^{ext} and z are 90° out of phase, giving a large $\text{Im}(z/F^{ext})$ and thus a peak in the PSD. The generalize PSD is [8]:

$$z_{PSD}^n(f) = \frac{2k_B T / Q_n}{\pi \kappa_n f_{on} \left[\left(1 - f^2 / f_{on}^2\right)^2 + (f / f_{on} Q_n)^2 \right]} \quad (9)$$

and $F_{PSD}^n(f) = z_{PSD}^n(f) \kappa_n^2$. For $f \rightarrow 0$, $z_{PSD}^n(f) = 2k_B T / (\pi \kappa_n f_{on} Q_n)$. Table I relates the generalized variables to the experimental variables. Here f_{on} is the resonance frequency; η is the viscosity, A_o and d are the surface area and gap size of the cylindrical shell viscous damper as shown; R and C are the junction resistance and capacitance. The inertia force for helium in a Helmholtz oscillator is provide by the diaphragm with a spring constant κ_d and area A_d . Although the Q of the Helmholtz oscillator reported in Ref. [1] is ~ 1 , the dissipative mechanism responsible for such a low Q is not yet

identified. It is also not known if the new dissipative mechanisms reported in superfluid ^3He system [9] are present here. To check consistency, we integrated Eq. (9) and showed that $\langle \Delta z_n^2 \rangle = \int_0^\infty z_{PSD}^n df = k_B T / \kappa_n$ as expected [10], providing another way to derive Eq.

(5). At low frequencies we also show that the current fluctuations for superconductor reduces to the familiar Johnson current noise. Finally, the PSD of ϕ can be used to determine the fundamental limit on the resolution of a superfluid gyroscope [11], and may point out ways to reduce its noise.

We would like to acknowledge helpful discussion with Dr. Christian Lindensmith, Dr. Yury Mukharsky, Dr. David Pearson and Prof. David Goodstein. This work was carried out at the Jet Propulsion Laboratory, California Institute of Technology, under a contract with the National Aeronautics and Space Administration.

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FIGURE CAPTIONS

Figure 1: A plot of $P_r(\phi_e, \delta\phi)$ as a function of ϕ . Solid line is for $\|E_J\|/k_B T = 40.63$, two values: $\phi_e = 0$ and $\phi_e = \phi_{\max} = 0.35\pi$ are plotted. Dashed line is for $\|E_J\|/k_B T = 1.172$, two values: $\phi_e = 0$ and $\phi_e = \phi_{\max} = 0.1\pi$ are plotted. The inset is a rigid pendulum model of the Josephson junction.

Figure 2: A plot of $\|E_J\|/k_B T$ as a function of ϕ_{\max} for a critical probability density of $P_c = 0.1/\text{radian}$ at $\delta\phi = \pi - 2\phi_{\max}$. The inset is a spring-mass oscillator model of the Josephson junction driven to oscillate at a small amplitude.

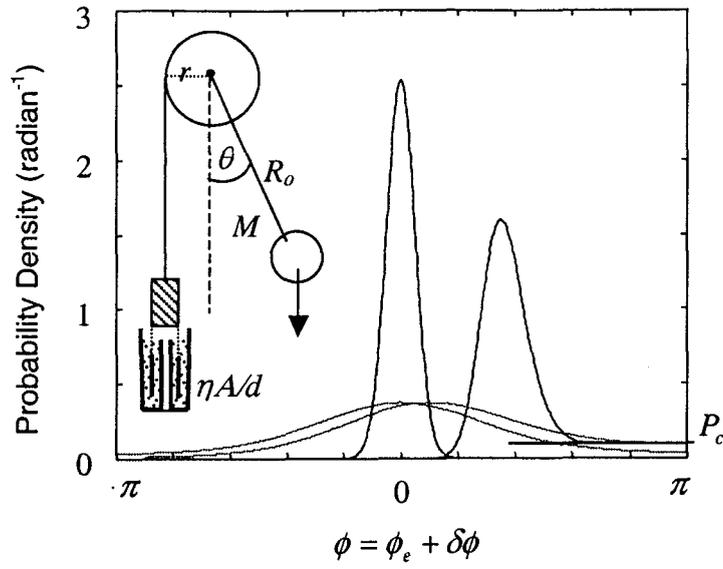


Figure 1

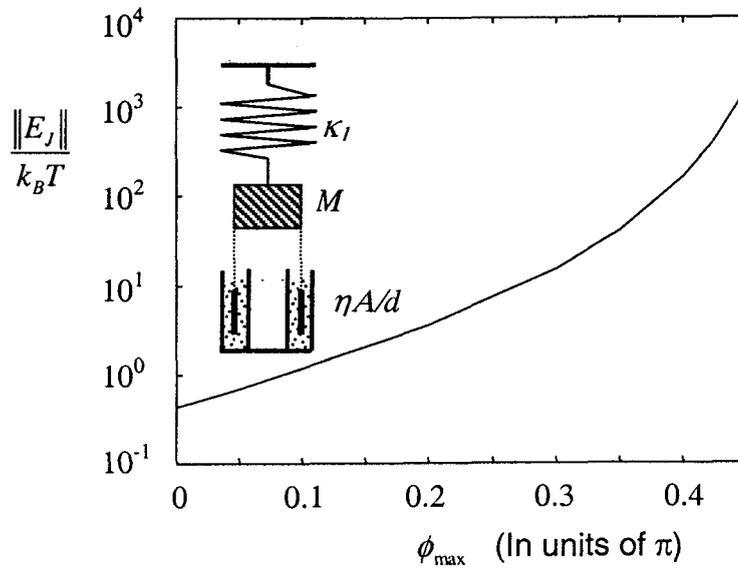


Figure 2

Table I: Relation between generalized variables and experimental variables.

Variables	Spring-Mass	Rigid Pendulum	Superconductor	Superfluid
z_n	z_1	$z_2 = \tilde{\theta}$	$z_3 = \phi$	$z_4 = \phi$
F_n	F_1	$F_2 = \Gamma$ $= MgR_o \sin \theta_e$	$F_3 = (\hbar/2e)I_c'' \sin \phi$	$F_4 = (\hbar/m)I_c \sin \phi$
F_n^{ext}	$F_1^{ext} = F_o e^{i\omega t}$	$F_2^{ext} = \tilde{\Gamma}$	$F_3^{ext} = (\hbar/2e)\tilde{I}''$	$F_4^{ext} = (\hbar/m)\tilde{I}$
M_n	M_1	$M_2 = MR_o^2$	$M_3 = \left(\frac{C\hbar^2}{4e^2} \right)$	$M_4 = \left(\frac{\rho^2 A_d^2 \hbar^2}{m^2 \kappa_d} \right)$
κ_n	$\kappa_1 = dF_1/dz_1$	$\kappa_2 = d\Gamma/d\theta$ $= MgR_o \cos \theta_e$	$\kappa_3 = dF_3/d\phi$ $= (\hbar/2e)I_c'' \cos \phi$	$\kappa_4 = dF_4/d\phi$ $= (\hbar/m)I_c \cos \phi$
ω_{on}^2	$\omega_{o1}^2 = \kappa_1/M_1$	$\omega_{o2}^2 = \kappa_2/M_2$ $= (g/R_o) \cos \theta_e$	$\omega_{o3}^2 = \kappa_3/M_3$ $= (2e/\hbar C)I_c'' \cos \phi$	$\omega_{o4}^2 = \frac{m\kappa_d I_c \cos \phi}{\hbar \rho^2 A_d^2}$
γ_n	$\eta A_o / (M_1 d)$	$\eta A_o r^2 / (M_1 d R_o^2)$	$\gamma_3 = \frac{1}{RC}$?
Q_n	ω_{o1}/γ_1	ω_{o2}/γ_2	$\sqrt{(2eR^2 C/\hbar)I_c'' \cos \phi}$	ω_{o4}/γ_4