Periodic-finite-type shift spaces

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Abstract — We introduce the class of periodic-finite-type (PFT) shift spaces. They are the subclass of shift spaces defined by a finite set of periodically forbidden words. Examples of PFT shifts arise naturally in the context of distance-enhancing codes for partial-response channels. We show that the class of PFT shifts represent a proper superset of the finite-type shift spaces and a proper subset of almost-finite-type shift spaces. We prove several properties of labeled graphs that present PFT shifts. For a given PFT shift space, we identify a finite set of forbidden words – referred to as "periodic first offenders" – that define the shift space and that satisfy certain minimality properties. Finally, we present an efficient algorithm for constructing labeled graphs that present PFT shift spaces.

I. INTRODUCTION

Magnetic recording systems often make use of binary codes that disallow the appearance of certain sequences that are problematic in the data recording or retrieval process. The set of allowable code sequences are generated by paths in a labeled, directed graph. Such sets of constrained sequences are referred to as sofic shift spaces in the symbolic dynamic literature and form a subset of the class of shift spaces.

Recently, several distance-enhancing constrained codes have been introduced that forbid the appearance of certain patterns in a periodic manner. In this paper, we examine properties of sets of sequences that satisfy such time-varying constraints, and establish their relationship to, and position within, the more familiar class of shift-invariant code constraints.

II. PERIODIC-FINITE-TYPE SEQUENCE SPACES

We consider bi-infinite sequences of symbols drawn from a finite alphabet $\mathcal{A}$, denoted $x = \ldots x_{-3} x_{-2} x_{-1} x_0 x_1 x_2 \ldots$ where each $x_i \in \mathcal{A}$. A finite block of symbols is referred to as a *word*.

Let $\mathcal{F}$ be a finite collection of words over $\mathcal{A}$ where each $w_j \in \mathcal{F}$ is associated with a non-negative integer index n_j. We write $\mathcal{F} = \left\{ w_1^{(n_1)}, w_2^{(n_2)}, \ldots, w_{|\mathcal{F}|}^{(n_{|\mathcal{F}|})} \right\}$ and associate with the indexed list $\mathcal{F}$ a *period* T, where T is a positive integer satisfying $T \geq \max\{n_1, n_2, \ldots, n_{|\mathcal{F}|}\} + 1$.

For a pair $\{\mathcal{F}, T\}$ and alphabet $\mathcal{A}$, let the periodic-finite-type sequence space $\mathsf{X}_{\{\mathcal{F},T\}}$ be the set of bi-infinite sequences that can be shifted such that the shifted sequence does not contain a word $w_j^{(n_j)} \in \mathcal{F}$ starting at any index m with $m \bmod T = n_j$.

III. PROPERTIES OF PERIODIC-FINITE-TYPE SPACES

We first show that every periodic-finite-type sequence space is a sofic shift space. Namely, that it may be presented by a labelled, directed graph. This implies the spaces are a subclass of shift spaces, hence we will say a shift space X is a shift of *periodic-finite-type* if there exists a pair $\{\mathcal{F}, T\}$ with $|\mathcal{F}|$ and T finite such that $X = \mathsf{X}_{\{\mathcal{F},T\}}$.

We distinguish X to be a *proper* periodic-finite-type shift space if X is periodic-finite-type but there is no pair $\{\mathcal{F}, T\}$ with $|\mathcal{F}|$ finite and $T = 1$ such that $X = \mathsf{X}_{\{\mathcal{F},T\}}$. The proper periodic-finite-type shifts are those which have a non-trivial description as a finite indexed forbidden list.

How can one determine if a graph presents a periodic-finite-type shift? We present three theorems that address this question. The first gives a necessary condition for an irreducible sofic shift to be a proper periodic-finite-type shift. The second gives a sufficient condition for an irreducible sofic shift to be periodic-finite-type, and a method to determine a corresponding periodic forbidden list of words. The third is a converse to the second.

We show that the irreducible periodic-finite-type shift spaces are almost-finite-type. As the sliding block coding theorem holds for almost-finite-type systems, this shows that there exist finite-state codes into irreducible periodic-finite-type shifts at rates less than or equal the Shannon capacity of the shift.

IV. MINIMAL DESCRIPTION

A shift space may be represented by various forbidden lists. It is useful to have a unique, minimal forbidden list which describes a shift space. Such a minimal description is well known for general shift spaces.

We extend the notion of a minimal description to periodic lists, identifying a finite set of forbidden words that define the space and are the unique minimal forbidden word description of the space for the given period.

V. GRAPH CONSTRUCTION ALGORITHM

We have developed a method to construct a graph whose complexity grows linearly with the length of the longest word and the number of words in the list. The algorithm is a generalization of the procedure from [1] to construct graphs presenting finite-type shift spaces. An alternative construction method for periodic-finite-type shifts may be found in [2].

REFERENCES

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