A cascade model of wave turbulence with applications to surface gravity and capillary waves

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ABSTRACT

A heuristic approach to the derivation of power spectra of wave motion is described and applied to capillary waves. The case of gravity waves studied earlier is briefly reviewed. In contrast to the previous studies, the nonlinearity of the wave motion is not required to be small, and the mean number of resonantly interacting wave harmonics is not limited to a smallest possible number (which is 4 for gravity waves on a deep fluid and 3 for capillary waves). The main external parameter of the problem is the input flux Q of the wave energy related to the mean wind velocity as $Q \propto U^3$. Depending on its value, wave spectra $S(\omega) \propto \omega^{-q}$ and F(k)= k^{-p} take various forms - from that corresponding to the weak-turbulence limit and to that corresponding to the saturated (Phillips') wave spectra, Fractal dimensions of the sea surface arc related to external factors, specifically to the mean wind speed. The theoretically predicted spectra are in good qualitative agreement with the observations on capillary waves conducted in a large wave tank.

I. Introduction

The subject of this paper is turbulence of surface gravity and capillary waves, although the formalism presented here can be applied to other oceanic and atmospheric wave motions, such as internal waves, Rossby waves, etc. The nonlinearity of these wave systems tends to be weak; hence, the corresponding problems are amenable to a treatment by small perturbation techniques [Zakharov et al.; 1992]. Unfortunately, such techniques do not account for highly nonlinear local processes associated with intermittently occurring steep or breaking waves. While the strongly nonlinear events are rare, they may have an appreciable effect on the overall energy transfer, hence on the spectra of wave turbulence. In contrast to the previous work, we employ here a heuristic approach which goes beyond the limit of weak nonlinearity. The well-known results of the weak turbulence theory (WTT) are derived as a special case.

The most important accomplishment of WTT is its prediction of power-law spectra for fields' oscillations in space and time. For example, the wavenumber spectrum for spatial variation of a fluids surface, $\eta(x;t)$, on scales of gravity waves is given by [Zakharov and Filinenko, 1966]:

$$F_\eta(k) = \alpha_g g^{-1/2} Q^{1/3} k^{-7/2} \tag{1.1}$$

in the range of capillary waves, the spectrum is [Zakharov and Filonenko, 1967]:

$$F_\eta(k) = \alpha_c \sigma^{-3/4} Q^{1/2} k^{-19/4} \tag{1.2}$$

where a_g and α_c are nondimensional universal ("Kolmogorov") constants, Q is the rate of energy transfer down the spectrum (divided by the fluid density, p), g *is* the acceleration of gravity, and σ is the coefficient of surface tension divided p. The angular, Θ, dependence of these two-dimensional spectra is omitted for simplicity.

The spectral energy transfer can be envisaged as a cascade process which manifests itself in the surface topography [Glazman and Weichman, 1989] or in the surface temporal oscillations: wavelets of a continuously decreasing size ride on top of a large-scale wave form. As a result, power spectra of surface height variations tend to be rather broad and, at frequencies well above the spectral peak frequency, they exhibit a power-law behavior. For a wavenumber spectrum of form $F(k) \propto k\text{-}p$, the exponent, p, is related to the Hausdorff (fractal) dimension, D_H, of the surface elevation field: $D_H = (8\text{-}p)/2$, which affords an instructive interpretation of the quantities involved. In particular, wave spectra can be conveniently presented as [Glazman and Weichman, 1989; Glazman, 1990]

$$F(k) \propto k^{-4+2\mu} \tag{1.4}$$

The value $\mu = 1/4$ corresponds to (1. 1), while $\mu = 0$ yields Phillips' saturation spectra discussed below. The meaning of μ follows from the relationship

$$D_H = 2 + \mu \tag{1.5}$$

Hence, μ represents the "fractal index," This quantity invariably emerges in analytical expressions for various geometrical properties of a random surface, such as its mean Gaussian curvature, root-mean-square slope, etc. [e.g., Glazman, 1990].

Apparently, the surface geometry characterized by (1 .2) is not fractal because the spectrum rolls off too rapidly giving $\mu < 0$. However, as shown in section 2, the surface's temporal oscillations due to capillary waves, recorded at a fixed location, represent a fractal process. In section 2 the range of fractal dimensions for $\zeta(t)$ with an arbitrary degree of nonlinearity is estimated and D_H *is* related to external parameters of the problem.

At a sufficiently high external input of the wave energy, both the capillary and the gravity wave turbulence becomes strong, and WTT is not applicable. The limiting spectra of strongl y nonlinear wave fields are known as "saturated" spectra [Phillips, 1977]. They are derived on the assumption that the energy transfer down the spectrum occurs primarily due to intermittent events of wave breaking rather than due to a Kolmogorov-type inertial cascade. The latter is too slow to support a high rate of energy input, while the breaking of (intermittently occurring sufficiently steep) waves allows the energy to be transferred to small scales by sporadic "jumps" from much larger scales. Respectively, the energy flux Q must not emerge in expressions for the wave spectra, and the limiting form is

$$F(k) = \beta k^{-4}, \tag{1.6}$$

- as based on analysis of dimensions [Phillips, 1958]. This is valid for both capillary and gravity waves [Phillips, 1977], and β is known as the Phillips constant,

Experimental observations consistently show that the actual spectra of surface waves, while exhibiting a power law behavior (1 .4), are not limited to the forms (1.1),(1.2) and (1.6): the exponent (hence, μ in (1 .4)) spans the entire range - from the weak turbulence limit (1.1),(1 .2) to the saturated spectrum limit (1 .6), - depending on the external conditions (the energy input and its partition between the component going down the spectrum and that consumed by other mechanisms of the total energy budget - see, e.g., [Glazman, 1993]). In what follows we suggest a simple heuristic theory explaining this behavior, and for the capillary wave spectrum we also provide comparison with laboratory measurements.

2. Multiwave interaction theory for surface gravity and capillary waves

The potential energy of the wave motion includes two components:

$$U = \tfrac{1}{2}\rho g \int \eta^2 dx + \sigma \int (\sqrt{1 + \nabla\eta^2} - 1) dx \tag{2.1}$$

The first one is due to the gravity force and the second due to the surface tension. Here, $\eta = \eta(x,t)$ is the elevation of the fluid surface above the zero-mean level. An approximate equi-partition of energy between the kinetic and the potential parts allows one to approximate the surface density of the total wave energy, E, by

$$E_g \approx \rho g \eta^2 \tag{2.2a}$$

$$E_\sigma \approx \sigma (\nabla\eta)^2 \tag{2.2b}$$

where component E_g is due to gravity waves and E_σ is due to capillary waves. These quantities are related to the spectral density of the wave energy by:

$$E = \int S(\omega) d\omega = \int\int F(k,\theta) k d\theta dk = \int G(k) dk\ , \tag{2.3}$$

where the integration is carried out over all wavenumbers/frequencies. Here, $S(\omega)$ is the frequency spectrum, $F(k,\theta)$ is the two-dimensional wavenumber spectrum and $G(k)$ is the wavenumber modulus spectrum of the wave energy. Obviously, for gravity waves, the relationship bet ween the energy spectrum, F of *(2.3)*, and the surface height spectrum, F_η of (1.1), is: $F = \rho g F_\eta$. For capillary waves, this relationship is: $F = \sigma k^2 F_\eta$.

Inertial flux of wave energy can be envisaged as a cascade process in the frequency space: at each n -th step of the cascade, the amount of energy, e_n, transferred from the previous step is estimated as:

$$e_n \approx \int_{\omega_{n-1}}^{\omega_n} S(\omega)d\omega = \int_{k_{n-1}}^{k_n} G(k)dk \tag{2.4}$$

where (ω_{n-1}, ω_n) is the width of a cascade step (which is much smaller than the width of the inertial range). Suppose, the characteristic time (the "turnover time") of nonlinear wave-wave interaction is t_n. Then, the rate Q of energy transfer through the spectrum is given by

$$Q = e_n / t_n \tag{2.5}$$

For both gravity and capillary waves, wc assume that the energy flux down the spectrum remains constant, hence Q is independent of k and ω. (A more complicated case of a non-conservative cascade is treated in [Glazman, 1992]). Provided e_n and t_n can be expressed in terms of k, ω and wave amplitude a, equation (2.5) allows one to derive the spectrum by means of elementary algebra (e.g., [Frisch et al., 1978]). Hence, we need to express these parameters in terms of the relevant quantities.

From (2.2a) it is obvious that en for gravity waves can be written as

$$e_n \approx \rho g a_n^2 \tag{2.6a}$$

For capillary waves, equation (2.2b) points to the following expression for e_n:

$$e_n \approx \sigma(a_n k_n)^2 \tag{2.6b}$$

Here, a_n is the Fourier amplitude of surface oscillation at the frequency/wavenumber scales ω_n and k_n, corresponding to the n -th step in the spectral cascade.

The derivation of the turnover time is formally based on the scaling of the collision integral in the kinetic equation [Zakharov and L'vov, 1975; Kitaigorodskii, 1983; Larazza et al, 1987] for the wave action spectral transfer. For the subsequent development it is useful to introduce the turnover time in a more general, although less formal, fashion. To this end we notice that the nonlinearity of wave processes is measured by the ratio, ε, of the fluid particle velocity, u, to the wave phase velocity, ω/k [Witham, 1974]. Since fluid particles in a surface wave execute an approximately orbital motion in the vertical plane with the radius equal to the wave amplitude and the period $2\pi/\omega$, the value of u at a given scale is found as $a_n\omega_n$. Respectively, the ratio $u/(c/\omega)$ is

$$\varepsilon_n \equiv \frac{u_n}{\omega_n / k_n} \approx \frac{a_n\omega_n}{\omega_n / k_n} = a_n k_n \tag{2.7}$$

This quantity represents the small parameter in deterministic perturbation theories. However, since the kinetic equation describing wave action, N(k) $=F(k)/\omega$, (or wave energy, $F(k)$) spectral transfer is derived for second statistical moments of the fields, the perturbation equations of statistical theory are developed in powers of ε^2. In such equations, terms (i.e., collision integrals) of order ε^2 correspond to three-wave interactions, terms ε^4 correspond to four-wave interactions [Zakharov et al., 1992], etc. in general, each additional Fourier component accounted for in the interaction integral adds new terms which are ε^2 times as great as a preceding term. The v-th term is of order $\varepsilon^{2(\nu-2)}$. Respectively, the characteristic time of nonlinear wave-wave interactions increases as the number of interacting harmonics grows. For 3-wave interactions, this time is estimated as $t^{-1} \approx \omega\varepsilon^2$, and for an arbitrary number, v, wc have

$$t^{-1} \approx \omega\varepsilon^{2(\nu-2)} \tag{2.8}$$

Formidable mathematical difficulties limit the kinetic equation to accounting only for the lowest-order resonant interactions. However, one can formall y present it as

$$\partial N / \partial t + \nabla_k \bullet \mathbf{T(k)} = \mathbf{p(k)}\ , \tag{2.9}$$

where ***p(k)*** is the spectral density of the input flux of wave action (from wind), and ∇_k . 7'(k) denotes the spectral density of the action flux due to all wave-wave interactions to order **v:**

$$\nabla_k \bullet \mathbf{7(k)} = I_3 + I_4 + \ldots + I_\nu\ , \tag{2.10}$$

I_ν arc collision integrals accounting for interactions among v waves satisfying resonance conditions

$$\begin{gathered} \omega_0 \pm \omega_1 \pm \ldots \pm \omega_\nu = 0 \\ \mathbf{k}_0 \pm \mathbf{k}_1 \pm \ldots \pm \mathbf{k}_\nu = 0 \end{gathered} \tag{2.11}$$

(non-resonant terms can be eliminated by appropriate canonical transformations [Zakharov et al., 1992]). For gravit y waves, the minimum number of resonantly interacting components is 4, while for capillary waves it is 3.

It has been argued earlier [Glazman, 1992] that intermittently occurring, rare events of steep and breaking waves (characterized by a locally high nonlinearity, hence a large, or even infinite, number of interacting Fourier components forming individual cusp-like wavelets), result in an increased mean (over a large time interval and large surface area) value of v. While this v may be substantially greater than the minimum resonant number appearing in WIT, the energy and action transfer may still be dominated by the weakly nonlinear inertial cascade. Thus, the "effective" v is introduced as an unknown function of the problem, the assumption of locality of wave-wave interactions in the wavenumber space remaining in force. Let us notice that the turnover time given by (2.8) for the highest-order term in (2.10) is the" slowest of all the times for "partial" fluxes associated with individual terms in (2.10). Therefore, although the total flux of the wave action, $\nabla_k \bullet$ T(k), is comprised of many partial fluxes I_3, I_4, etc., the appropriate characterist ic time for the integral transfer is given by (2.8).

We consider the case when the external input is concentrated at wavenumbers below certain k_0 marking the high-wavenumber boundary of the "generation range." Therefore, at $k > k_0 : p(k) = 0$, and a spectral flux is purely inertial. obviously, the energy flux down the spectrum is given by

$$Q = \int_0^{k_0} \omega(k) k dk \int_{-\pi}^{\pi} p(k,\theta) d\theta \qquad (2.12)$$

Respectively, equation (2.9) for the inertial range corresponds to

$$e_n \omega_n (a_n k_n)^{2(\nu-2)} \approx Q \ (= const) \qquad (2.13)$$

where $n \geq 1$.

Using the dispersion relation for capillary waves, $\omega^2 = \sigma k^3$, and (2.6b), equation (2.1 3) yields $e_n \approx Q^{1/(\nu-1)} \sigma^{(\nu-2)/(\nu-1)} \omega^{-1/(\nu-1)}$. The frequency spectrum is obtained by noticing that (2.4) yields a scaling relationship: $S(\omega) \propto e_n / \omega$. Introducing a proportionality constant, α_c, playing the role of the Kolmogorov constant, we ultimately arrive at the energy spectrum for surface capillary waves:

$$S(\omega) = \alpha_c Q^{1/(\nu-1)} \sigma^{(\nu-2)/(\nu-1)} \omega^{-\text{“/(”..1)}} \qquad (2.14)$$

This is related to the frequency spectrum of the wave slope by $S_{\nabla\eta}(\omega) = \sigma^{-1} S(\omega)$, whereas the surface elevation spectrum is found as $S_\eta(\omega) = S_{\nabla\eta}(\omega) k^{-2}$, i.e.:

$$S_\eta(\omega) = \alpha_c Q^{1/(\nu-1)} \sigma^{(2\nu-5)/[3(\nu-1)]}{}_{a}\text{-}(7\nu-4)/[3(\nu-1)] \qquad (2.15)$$

The one-dimensional process described by (2.15) is fractal for all v23 : the Hausdorff dimension of $\eta(t)$ whose spectrum falls off as ω^{-q} is given by (sw, e.g., [Glazman a Wcichman, 1989]) $D_H = (5-q)/2$ which translates into

$$D_H = \frac{8\nu - 11}{6(\nu-1)} \qquad (2.16)$$

Therefore, $D_H > 1$ for all $\nu \geq 3$. The "Phillips spectrum" ($\nu \to 00$) takes the form

$$S_\eta(\omega) = \beta \sigma^{2/3} \omega^{-7/3} \qquad (2.17)$$

which yields D_H = 4/.? , while the Zakharov-Filonenko spectrum in the frequency domain, $S(\omega) \propto \omega^{-17/6}$, yields $D_H = 13/12$.

The surface elevation spectrum in the wavenumber domain (omitting the angular spread factor) is found as:

$$F_\eta(k) \quad = k^{-1} \left[S_\eta(\omega) \frac{d\omega}{dk} \right]_{\omega=\omega(k)} {}^{1/(\nu-1)} \sigma^{-9/[6(\nu-1)]}{}_c k^{-(8\nu-5)/[2(\nu-1)]} \qquad (2.18)$$

(Factor 3/2 is included into the Kolmogorov constant α_c'). This coincides with the Zakharov-Filonenko spectrum (1 .2) in a special case of $\nu=3$, and it produces the Phillips spectrum (1.6) at v $\rightarrow \infty$. In the latter case, the surface becomes marginally fractal, that is: $D_H = 2$.

It is also easy to show that the use of the gravity wave dispersion relation, $\omega^2 = kg$, and of (2.6a) in *(2.13)* yields spectra of surface gravity waves. in particular the frequency spectrum of wave energy is [Glazman, 1992]:

$$S(\omega) = \alpha_g Q^{1/(\nu-1)} (\rho g^3)^{(\nu-2)/(\nu-1)} \omega^{-(5\nu-8)/(\nu-1)} \tag{2.19}$$

This reduces to the Zakharov-Filonenko (1. 1) and Phillips (1.6) spectra at v = 4 and v $\rightarrow$ 00, respectively.

3. Determination of ν for the inertial range

Provided v is an increasing function of wind, (2.15) and (2.18) agree with the laboratory observations by Jähne and Riemer [1990] which show a monotonic decrease of the spectrum slope (in the capillary range) as the wind increases from about 3 to about 9 m/s. Therefore, wc now must prove that ν increases as the wind grows. In order to avoid considering the entire problem of wave generation by wind, let us assume both the energy flux, Q, arriving from the lower-fre.ucncy (gravity wave) range and the wave spectrum in that range to be known. Our task is to match that spectrum with (2.15) and (2. 18). Let ω_0 designate the lowest-frequency boundary of the capillary range at which $S_\eta(\omega_0)$ and Q arc specified based on the matching of gravity and capillary spectra at a certain frequency ω_0. Taking (2. 15) at $\omega = \omega_0$ and solving it for v yields:

$$\nu = a_0 + a_1 V \tag{3.1}$$

where

$$a_0 = \frac{3(Z_0 - A) - 5\Gamma + 4\Omega_0}{3(Z_0 - A) - 2\Gamma + 7\Omega_0}, \qquad a_1 = \frac{3}{3(Z_0 - A) - 2\Gamma + 7\Omega_0}$$

and

$$Z_0 = \log S_\eta(\omega_0),\ A = \log \alpha_c,\ \Gamma = \log\sigma,\ \Omega_0 = \log \omega_0 \text{ and } V = \log Q .$$

Assuming that the gravity wave spectrum obeys the Phillips law (1.6), both a_0 and a_1 become independent of wind. Therefore, since Q is proportional to the cube of wind velocity, (3.1) confirms a gradual (actually, logarithmic) increase of v with an increasing wind.

The Jähne and Riemer [1990] experiments were conducted in a wave tank with the wind fetch under 90 m. Under such conditions, the gravity wave spectrum is developed very poorly, hence it falls off at least as rapidly as (1.6). While additional experiments would be desirable to cover cases of better developed seas, the experiments by Jähne and Riemer provide an encouraging agreement with the above heuristic theory.

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