EXTENDED HORIZON LIFTINGS FOR STABLE INVERSION OF NONMINIMUM-PHASE SYSTEMS

David S. Bayard
Jet Propulsion Laboratory
California Institute of Technology
4800 Oak Grove Drive
Pasadena, CA 91109

ABSTRACT

The lifting of Lozano [9] is generalized, and shown to be one in a large class of liftings which enjoy the same zero annihilation properties (i.e., placement of transmission zeros to the origin). Many useful properties of the new liftings are proved, and the results are discussed relevant to recent problems in control theory (i.e., stable plant inverse control of nonminimum phase systems) and communications theory (i.e., stable equalization of nonminimum phase channels).

1. INTRODUCTION

In Lozano **[9]**, a multirate sampling method is presented which allows stable inversion of any linear time-invariant finite-order plant, regardless of whether it is minimum phase or nonminimum phase. The general approach is based on the notion of a mathematical "lifting" in which a serial-to-parallel conversion is performed on the plant input and output signals, and mappings are considered between the vectorized quantities. The key property of Lozano's lifting which makes it so useful is that the transmission zeros of the lifted plant are annihilated (i. e., placed to the origin). This zero annihilation (ZA) property allows the lifted plant to be stably inverted using standard control methods. Not surprisingly, this lifting has also been applied to developing stable adaptive control algorithms for nonminimum phase systems (cf., [3][4][9][10][11]).

Since Lozano's lifting utilizes a horizon size of $2n$ where n is the plant order, it will be denoted as the "$2n$-lifting". In this paper, Lozano's $2n$-lifting is generalized, and shown to be one in a large class of liftings which enjoy the same zero annihilation properties. Unlike the $2n$-lifting, the generalized class of liftings allows the use of extended horizons (i.e., horizons larger than $2n$). An important consequence is that extended horizon liftings lead to plant-inverse controllers with significantly reduced control gains. This overcomes a bottleneck associated with the $2n$-lifting which has prevented its use in many applications of practical interest. A simulation example is provided in which *the peak control requirement is reduced by four orders of magnitude* using an extended horizon approach.

As a dual result, it is shown that a related class of liftings enables equalization of nonminimum phase channels in communication systems. This overcomes the standard bottleneck of inverting the channel in a stable fashion. In this case, the extended horizon property allows channel inversion by least squares estimation, which provides smoothing in the case of noise.

2. BACKGROUND AND NOTATION

Consider the input/output model,

$$\mathcal{A}(z^{-1})y_t = \mathcal{B}(z^{-1})u_t \tag{2.1a}$$

$$\mathcal{A}(z^{-1}) = 1 + \sum_{i=1}^{n} a_i z^{-i}; \qquad \mathcal{B}(z^{-1}) = \sum_{i=1}^{n} b_i z^{-i} \tag{2.1b}$$

where polynomials $\mathcal{A}$ and $\mathcal{B}$ are assumed to be relatively prime. It is assumed that $b_1 \neq 0$, so that the polynomial $\mathcal{B}$ can be factored uniquely into the form $\mathcal{B}(z^{-1}) = z^{-d} b_1 \bar{\mathcal{B}}(z^{-1})$ where $\bar{\mathcal{B}}(z^{-1})$ is monic and $d = 1$ is the plant delay. The choice $d = 1$ is for convenience only and is not a fundamental restriction. In the case that $d \neq 1$, all subsequent expressions can be appropriately modified without loss of generality.

Choose some horizon time $N > \bar{n}$. The system (2.1) is iterated to give the following system of linear equations,

$$Y(k) = A_1 Y(k) + A_2 Y(k-1) + B_1 U(k) + B_2 U(k-1) \tag{2.2}$$

where,

$$Y(k) = \begin{bmatrix} y_{kN+1} \\ y_{kN+2} \\ \vdots \\ y_{kN+N} \end{bmatrix}; \qquad U(k) = \begin{bmatrix} u_{kN} \\ u_{kN+1} \\ \vdots \\ u_{kN+N-1} \end{bmatrix} \tag{2.3}$$

A_1 = lower triangular Toeplitz, with first column $[0, -a_1, \ldots, -a_n, 0, \ldots, 0]^T$

A_2 = upper triangular Toeplitz, with first row $[0, \ldots 0, -a_n, \ldots, -a_1]$

B_1 = lower triangular Toeplitz, with first column $[b_1, b_2, \ldots, b_n, 0, \ldots 0]^T$

B_2 = upper triangular Toeplitz, with first row $[0, \ldots 0, b_n, \ldots, b_2]$

Example 1 Let $n = 3$ and $N = 4$. Then, (2.2) becomes,

$$\begin{bmatrix} y_{4k+1} \\ y_{4k+2} \\ y_{4k+3} \\ y_{4k+4} \end{bmatrix} = \left[\begin{array}{cccc|cccc} 0 & -a_3 & -a_2 & -a_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_3 & -a_2 & -a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_3 & -a_2 & -a_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_3 & -a_2 & -a_1 & 0 \end{array}\right] \left[\begin{array}{c} y_{4k-3} \\ y_{4k-2} \\ y_{4k-1} \\ y_{4k} \\ \hline y_{4k+1} \\ y_{4k+2} \\ y_{4k+3} \\ y_{4k+4} \end{array}\right]$$

$$
+\left[\begin{array}{cccc|cccc}
0 & 0 & b_3 & b_2 & b_1 & 0 & 0 & 0 \\
0 & 0 & 0 & b_3 & b_2 & b_1 & 0 & 0 \\
0 & 0 & 0 & 0 & b_3 & b_2 & b_1 & 0 \\
0 & 0 & 0 & 0 & 0 & b_3 & b_2 & b_1
\end{array}\right]
\left[\begin{array}{c}
u_{4k-4} \\ u_{4k-3} \\ u_{4k-2} \\ u_{4k-1} \\ \hline u_{4k} \\ u_{4k+1} \\ u_{4k+2} \\ u_{4k+3}
\end{array}\right]
$$

It is convenient to combine terms involving $Y(k)$ in (2.2) and rearrange to give the following lifting of Albertos [1],

Albertos' Lifting:

$$Y(k) = AY(k-1) + HU(k) + BU(k-1) \tag{2.4}$$

where,

$$A = (I - A_1)^{-1} A_2 \tag{2.5a}$$

$$H = (I - A_1)^{-1} B_1 \tag{2.5b}$$

$$B = (I - A_1)^{-1} B_2 \tag{2.5c}$$

It is noted that since A_1 is lower triangular with zeros on the diagonal, the quantity $(I - A_1)$ is always invertible. Hence the quantities in (2.5) always exist.

Polynomial A is divided into B to give impulse response sequence $\{ h_i \}$,

$$\frac{B(z^{-1})}{A(z^{-1})} = \sum_{i=1}^{\infty} h_i z^{-i} \tag{2.6}$$

The Markov parameter sequence $\{ h_i \}$ is not assumed to be convergent (i.e., the system may be unstable). Using the Toeplitz structure of A_1 and B_1 and relation (2.6), it can be shown [1][3] that the matrix H in (2.4) (2.5b) can be written in terms of the impulse response parameters,

$$H = \begin{bmatrix} h_1 & 0 & \cdots & 0 \\ h_2 & h_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ h_N & \cdots & h_2 & h_1 \end{bmatrix} \tag{2.7}$$

This is the desired expression for H, i.e.,

$H =$ lower triangular Toeplitz, with first column $[h_1, h_2, \ldots, h_N]^T$

Since the delay is unity by assumption (i.e., d = 1), the matrix H has a nonzero diagonal (i.e., $h_1 \neq 0$), and is always invertible.

3. GENERALIZED LIFTINGS

In this section, a new class of liftings will be defined by generalizing the lifting of Albertos (2.4). For this purpose, it will be useful to construct the "small" vector $Y_s(k)$ from $Y(k)$ as follows,

$$Y_s(k) \triangleq S_y Y(k) \in R^{\sigma_y}$$

where $S_y \in R^{\sigma_y \times N}$ is a *selection* matrix which sifts out σ_y elements of $Y(k)$ for inclusion into $Y_s(k)$.

The matrix S_y is most conveniently constructed from a 0-1 vector ρ_y. For example, if $\rho_y = [1,1,0,0]$ then $Y_s(k) \in R^2$ contains the first two elements of $Y(k) \in R^4$ and the selection matrix is given as,

$$S_y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

A systematic method to construct S_y is defined as follows: *form a diagonal matrix from the entries of vector ρ_y, and then remove all rows made up entirely of 0s.* This construction defines the mapping $\mathcal{W} : R^N \rightarrow R^{\sigma_y \times N}$ for which one can write $S_y = \mathcal{W}(\rho_y)$. Since ρ_y can be uniquely reconstructed by a logical "or" over the columns of S_y, the mapping $\mathcal{W}$ is one-to-one.

Using the above notation, the following "small" vectors are defined,

$$Y_s(k) \triangleq S_y Y(k); \quad S_y \triangleq \mathcal{W}(\rho_y) \in R^{\sigma_y \times N} \tag{3.1a}$$

$$U_s(k) \triangleq S_u U(k); \quad S_u \triangleq \mathcal{W}(\rho_u) \in R^{\sigma_u \times N} \tag{3.1b}$$

$$Y_s^c(k) \triangleq S_y^c Y(k); \quad S_y^c \triangleq \mathcal{W}(\rho_y^c) \in R^{(N-\sigma_y)\times N} \tag{3.1c}$$

where ρ_y and ρ_u are specified 0-1 window vectors, and ρ_y^c is defined as the 0-1 *complement* of ρ_y.

The vector $Y_s^c(k)$ in (3.1c) is denoted as the *complementary output* since it is comprised of all elements of the vector $Y(k)$ which are *not included* in $Y_s(k)$. A formula to reconstruct $Y(k)$ from $Y_s(k)$ and $Y_s^c(k)$ is now derived. It is noted that the quantity $\bar{S}_y = [S_y^T, (S_y^c)^T]^T$ is a permutation matrix. Hence its inverse is its transpose, i.e., $\bar{S}_y^T \bar{S}_y = I$, which gives upon expanding,

$$S_y^T S_y + (S_y^c)^T S_y^c = I \tag{3.2a}$$

Multiplying both sides of (3.2a) on the right by $Y(k)$ gives the deisred formula,

$$Y(k) = S_y^T Y_s(k) + (S_y^c)^T Y_s^c(k) \tag{3.2b}$$

Example 2 Fig. 1 graphically depicts partial horizon vectors U_s and Y_s for the case $N = 6$, as determined by window vectors $\rho_u = [0,1,1,1,0,0]$ and $\rho_y = [0,0,1,1,1,0]$. In this case, $\rho_y^c = [1,1,0,0,0,1]$ and one can compute,

$$S_u = \mathcal{W}(\rho_u) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}; \quad S_y = \mathcal{W}(\rho_y) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$S_y^c = \mathcal{W}(\rho_y^c) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

■

Using the notation developed above, a new family of liftings will be defined by generalizing the lifting of Albertos (2.4), *As a key step, it will be assumed that $U(k)$ is chosen as zero outside the window defined by ρ_u*. Mathematically this can be written as,

$$(I - S_u^T S_u)U(k) = 0 \tag{3.3}$$

Consider the following nomminimal state-spacw realization of the Albertos lifting (2.4) determined using the small vectors in (3.1) and identity (3.3),

$$\begin{bmatrix} Y(k) \\ U_s(k) \end{bmatrix} = \begin{bmatrix} A & BS_u^T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Y(k-1) \\ U_s(k-1) \end{bmatrix} + \begin{bmatrix} HS_u^T \\ I \end{bmatrix} U_s(k) \tag{3.4}$$

$$Y_s(k) = [\,S_y \quad 0\,] \begin{bmatrix} Y(k) \\ U_s(k) \end{bmatrix} \tag{3.5(7)}$$

Substituting for the state in the output equation (3.5a) gives the alternative output equation,

$$Y_s(k) = [\,S_y A \quad S_y B S_u^T\,] \begin{bmatrix} Y(k-1) \\ U_s(k-1) \end{bmatrix} + S_y H S_u^T U_s(k) \tag{3.5b}$$

For analysis purposes, it is convenient to transform the open-loop plant using the similarity transformation T where,

$$\begin{bmatrix} Y_s(k) \\ Y_s^c(k) \\ U_s(k) \end{bmatrix} = T \begin{bmatrix} Y(k) \\ U_s(k) \end{bmatrix} \tag{3.6}$$

$$T = \begin{bmatrix} C \\ M \end{bmatrix}; \quad C = [\,S_y \quad 0\,]; \quad \mathbf{M} = \begin{bmatrix} S_y^c & 0 \\ 0 & I \end{bmatrix} \tag{3.7}$$

$$S_y^c \triangleq \mathcal{W}(\rho_y^c) \tag{3.8}$$

It can be verified that transformation T is square and invertible. Furthermore, T is in the form of a *permutation matrix* which reorders the state such that the components of Y_s appear first, and the remaining elements follow in the specified order. Since T is a permutation matrix, the inverse of T is given simply by its transpose (cf., Barnett [2], pp. 374), i.e.,

$$T^{-1} = T^T = [C^T, M^T] \tag{3.9}$$

Transforming the open-loop dynamics (3.4) by the similarity transformation (3.6)(3.7), gives rise to a very useful representation denoted as the Generalized Lifting System Model, *Generalized Lifting System Model, $\mathcal{G}(\rho_u, \rho_y)$:*

$$\begin{bmatrix} Y_s(k) \\ Y_s^c(k) \\ U_s(k) \end{bmatrix} = \begin{bmatrix} S_y A S_y^T & S_y A (S_y^c)^T & S_y B S_u^T \\ S_y^c A S_y^T & S_y^c A (S_y^c)^T & S_y^c B S_u^T \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Y_s(k-1) \\ Y_s^c(k-1) \\ U_s(k-1) \end{bmatrix} + \begin{bmatrix} S_y H S_u^T \\ S_y^c H S_u^T \\ 1 \end{bmatrix} U_s(k) \tag{3.10}$$

It is noted that the generalized lifting $\mathcal{G}(\rho_u, \rho_y)$ is defined uniquely by the choice of selection windows ρ_u and ρ_y, from which the matrices S_u, S_y, S_y^c are calculated. The Generalized Lifting system model (3.10) is depicted in the block diagram of Fig. 2. It is seen that Y_s and Y_s^c form two coupled subsystems which are driven by a common input U_s. It is also noted that the transmission zeros of the transfer function from $U_s(k)$ to $Y_s(k)$ are affected by the choice of windows ρ_u and ρ_y. This is a key feature which will be used to advantage in later developments.

The lifting $\mathcal{G}(\rho_u, \rho_y)$ generalizes a number of existing results. For example, the lifting (2.4) in Albertos [], is equivalent to the choice of windows,

Albertos' Lifting:

$$\rho_u = [\overbrace{1, 1, ..., 1, 1}^{N}] \tag{3.11a}$$

$$\rho_y = [\overbrace{1, 1, ..., 1, 1}^{N}] \tag{3.11b}$$

The $2n$-lifting in [9] is equivalent to the choice of windows,

Lozano's $2n$-Lifting:

$$\rho_u = [\overbrace{0, \overbrace{1, 1, ..., 1}^{n}, \overbrace{0, ..., 0}^{n-1}}^{N}] \tag{3.12a}$$

$$\rho_y = [0, \overbrace{0, ..., 0}^{n-1}, \overbrace{1, 1, ..., 1}^{n}] \tag{3.12b}$$

Since each choice of ρ_u and ρ_y gives rise to a unique lifting, there are a total of 2^{N+1} possible generalized liftings over a horizon of length N. Aside from (3.11) and (3.12), it appears that none of these new liftings have been investigated in the literature. It will be seen subsequently that many of these new liftings have very useful properties.

4. ZERO ANNIHILATION

In the previous section a new class of liftings was introduced. In this section, we will focus only on those liftings for which the transmission zeros of the (squared down) lifted system lie at the origin.

For notational convenience, we define the "small" matrix H_s by,

$$H_s = S_y H S_u^T \tag{4.1}$$

The quantity H_s appears in many expressions and will play an important role in subsequent proofs. The matrix H_s can be obtained directly from ρ_u and ρ_y by writing ρ_u along the top of H and ρ_y along the side of H. This arrangement is depicted graphically in Fig. 3. The matrix H_s is then the submatrix defined by the elements of H having 1's along both borders. Two properties of interest concerning H_s are,

Output Tracking (OT) Condition:

$$H_s H_s^\dagger = 1 \tag{4.2}$$

Input Tracking (IT) Condition:

$$H_s^\dagger H_s = 1 \tag{4.3}$$

where superscript "†" denotes the Moore-Penrose inverse. It is noted that both the OT and IT conditions are satisfied if H_s is square and invertible.

The property of placing transmission zeros of the lifted plant to the origin is characterized in the next result.

Lemma 1 (Zero Annihilation) *Assume that windows ρ_u and ρ_y satisfy,*

Zero Annihilation (ZA) conditions:

$$B S_u^T = 0 \tag{4.4a}$$

$$A (S_y^c)^T = 0 \tag{4.4b}$$

where,

$$S_u = \mathcal{W}(\rho_u);\ S_y = \mathcal{W}(\rho_y);\ S_y^c = \mathcal{W}(\rho_y^c)$$

Then,

(i) the generalized lifting (9.10) has the simplified representation (cf., Fig. 4),

$$Y_s(k) = S_y A S_y^T Y_s(k-1) + H_s U_s(k) \tag{4.5a}$$

$$Y_s^c(k) = S_y^c A S_y^T Y_s(k-1) + S_y^c H S_u^T U_s(k) \tag{4.5b}$$

(ii) Y_s^c is unobservable from Y_s and has stable (deadbeat) dynamics

Furthermore,

(iii) *If H_s in (4.1) is Square and invertible, then the transmission zeros of the lifted transfer function (3.10) from U_s to Y_s are annihilated (i. e., lie at the origin).*

(iv) *If the OT condition (4.2) is satisfied then the transmission zeros of the "squared down" lifted transfer function (3.10) from V (where $U_s = H_s^{\dagger}V$) to Y_s are annihilated.*

Proof: Results (i) and (ii) follow by substituting the ZA conditions (4.4) into (3.10), to give,

$$\begin{bmatrix} Y_s(k) \\ Y_s^c(k) \\ U_s(k) \end{bmatrix} = A_p \begin{bmatrix} Y_s(k-1) \\ Y_s^c(k-1) \\ U_s(k-1) \end{bmatrix} + B_p U_s(k) \tag{4.6}$$

$$Y_s(k) = C_p \begin{bmatrix} Y_s(k-1) \\ Y_s^c(k-1) \\ U_s(k-1) \end{bmatrix} + D_p U_s(k) \tag{4.7}$$

where,

$$A_p = \begin{bmatrix} S_y A S_y^T & 0 & 0 \\ S_y^c A S_y^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad B_p = \begin{bmatrix} S_y H S_u^T \\ S_y^c H S_u^T \\ I \end{bmatrix} \tag{4.8}$$

$$C_p = [\, S_y A S_y^T \quad 0 \quad 0 \,] \; ; \quad D_p = H_s \tag{4.9}$$

According to standard definitions (cf., Davison and Wang [8]) values of λ satisfying,

$$\det \begin{bmatrix} A_p - \lambda I & B_p \\ C_p & D_p \end{bmatrix} = 0 \tag{4.10}$$

are the *transmission zeros* of the transfer function from $U_s(k)$ to $Y_s(k)$ defined by the state-space model (A_p, B_p, C_p, D_p). Consider the following identity,

$$\det(G) = \det(G_{22}) \det(G_{11} - G_{12} G_{22}^{-1} G_{21}) \tag{4.11}$$

where, $G \triangleq \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$ and the inverse of G_{22} is assumed to exist. Assuming that H_s is invertible, the identity (4.11) can be applied to (4.10), which gives upon substituting (4.8)(4.9),

$$\det \begin{bmatrix} -\lambda \cdot I & 0 & 0 \\ X & -\lambda \cdot I & 0 \\ X & 0 & -\lambda \cdot I \end{bmatrix} = (-\lambda)^{N + \sigma_u}$$

which proves (iii). Result (iv) follows by an identical analysis assuming that the OT condition (4.2) holds, and that the simplified plant (A_p, B_p, C_p, D_p) in (4.6)-(4.9) has been squared down by a precompensator $H_s^{\dagger}$. ∎

Lemma 1 is important since it gives conditions which ρ_u and ρ_y must satisfy for the generalized lifting $\mathcal{G}(\rho_u, \rho_y)$ to have its transmission zeros at the origin. All results in Lemma 1 can be simply understood by comparing Fig. 3 with Fig. 4 and by noting all of the blocks that have vanished under the ZA conditions. It is seen that Y_s^c no longer couples into the Y_s subsystem. Furthermore, the Y_s^c subsystem has become deadbeat i.e., all of the poles of the Y_s^c subsystem are at the origin. Most importantly, there is now only one forward path from U_s to Y_s. Clearly something drastic has happened to the system zeros. Rigorously, if H_s is square invertible, result (iii) of Lemma 1 states that the transmission zeros of the transfer function from $U_s(k)$ to $Y_s(k)$ have been placed to the origin (i.e., annihilated). If H_s is not square but the or" condition holds, result (iv) of Lemma 1 states that the zeros the lifted plant "squared down" by a *precompensator* $H_s^{\dagger}$ are annihilated.

5. EXTENDED HORIZON LIFTINGS

The next result introduces a new class of liftings which satisfy the conditions of Lemma 1.

Theorem 1 *A class of generalized liftings $\mathcal{G}(\rho_u, \rho_y)$ which satisfy both the ZA and OT conditions is of the following form,*

Extended Horizon Lifting (OT Form):

$$\rho_u = \overbrace{[\overbrace{0,\ldots,0}^{m},\overbrace{1,\ldots,1}^{\ell},\overbrace{p}^{p},\overbrace{1,\ldots,1}^{n},\overbrace{0,\ldots,0}^{q},\overbrace{0,\ldots,0}^{n-1}]}^{N} \tag{5.1a}$$

$$\rho_y = [\overbrace{0,\ldots,0}^{m},\overbrace{0,\ldots,0}^{\ell},\overbrace{p}^{p},\overbrace{0,\ldots,0}^{q},\overbrace{0,\ldots,0}^{n-1},\overbrace{1,\ldots,1}^{n}] \tag{5.1b}$$

where $m \geq 0$ and $\ell \geq 0$ are arbitrary, $q = 0$, $p \in R^p$ is an arbitrary (or null) 0-1 vector chosen identically in both ρ_u and ρ_y; and $n > 0$ is the order of the irreducible plant (2.1). Furthermore, if the system (2.1) is obtained by a zero-order hold (ZOH) digitization of a continuous-time plant, the integer $q \geq 0$ can be chosen arbitrarily.

Proof: By definition, the matrix B in (2.5c) has the sparse form $B = [O_b|X_b]$ where $O_b \in R^{N \times N-n+1}$ is a matrix of all "0" elements, and $X_b \in R^{N \times n-1}$. By construction of S_u from ρ_u in (5.1a), the nonzero elements of S_u multiply only elements of O_b in the product BS_u^T. Hence $BS_u^T = 0$. Likewise, the matrix A in (2.5a) has the sparse form $A = [O_a|X_a]$ where $O_a \in R^{N-n}$ is a matrix of all "0" elements, and $X_a \in R^{N}$. By construction of S_y^c from ρ_y^c (i.e., 0-1 complement of ρ_y in (5.1b)), the nonzero elements of S_y^c multiply only elements of O_a in the product $A(S_y^c)^T$. Hence $A(S_y^c)^T = 0$ and the ZA conditions (4.4) are satisfied.

In order to show the OT condition, first consider the case where $q \geq 0$. Then choice (5.1) ensures that H_s is of the form (cf., Fig. 3),

$$H_s = \begin{array}{c c} & \begin{array}{ccc} \ell & p & n \end{array} \\ \begin{array}{c} p \\ n \end{array} & \begin{pmatrix} X & | & \mathcal{F} & 0 \\ X & | & X & \mathcal{H} \end{pmatrix} \end{array} \tag{5.2}$$

where $\mathcal{F} \in R^{p \times p}$ is lower triangular with a nonzero diagonal (and hence is invertible), and $\mathcal{H} \in R^{n \times n}$ is given by,

$$\mathcal{H} = \begin{bmatrix} h_{n+q} & \dots & h_{1+q} \\ \vdots & & \vdots \\ h_{2n+q-1} & \dots & h_{n+q} \end{bmatrix} \qquad (5.3)$$

Since $\mathcal{F}$ is full rank, it follows from the special structure of (5.2) that H_s is full rank if $\mathcal{H}$ has full rank. To show that $\mathcal{H}$ has full rank, let (A, b, c) be any minimal (i.e., controllable and observable) state-space realization of the transfer function (2.1). The Markov parameters $\{h_i\}$ can be written as $h_i = cA^{i-1}b$, $i = 1, \dots, \infty$. Substituting into (5.3) gives,

$$\mathcal{H} = \begin{bmatrix} cA^{n+q-1}b & \dots & cA^{q}b \\ \vdots & & \vdots \\ cA^{2n+q-2}b & \dots & cA^{n+q-1}b \end{bmatrix} \qquad (5.4)$$

$$= \mathcal{O}A^{q}\mathcal{C}\hat{I} \qquad (5.5)$$

where $\hat{I}$ is a reversed identity (i.e., $I = [e_1, \dots, e_n]$, $\hat{I} = [e_n, \dots, e_1]$), and $\mathcal{O}$ and $\mathcal{C}$ are observability and controllability matrices of (A, b, c), respectively. Since (A, b, c) is controllable and observable, it follows that $\mathcal{O}$ and $\mathcal{C}$ are each full rank. The fact the system (2.1) is obtained by a ZOH digitization implies that $A \in R^{n \times n}$ is full rank (i.e., A is a state-transition matrix). These facts together imply that $\mathcal{H}$ in (5.2) is full rank and hence H_s is invertible.

If the system (2.1) is not obtained by ZOH digitization, the matrix A may not be full rank. However, the results still hold with the restriction that $q = 0$ since in this case A^q is replaced by identity matrix I in relation (5.5), which is always full rank. ■

Theorem 1 is important because it generalizes Lozano's $2n$-lifting to a much larger class of liftings which enjoy the same zero annihilation properties. Note that the extended horizon liftings have total horizon length $N = m + \ell + p + q + 2n - 1$, which can be chosen longer than Lozano's lifting for which $N = 2n$. Hence the name "extended horizon". It will be seen that these extra degrees of freedom will overcome several difficulties associated with the $2n$-lifting.

The following result is essentially a "dual" to the previous theorem.

Theorem 2 *A class of generalized liftings $\mathcal{G}(\rho_u, \rho_y)$ which satisfy both the ZA and IT conditions is of the following form,*

Extended Horizon Lifting (IT Form):

$$\rho_u = \overbrace{[\underbrace{0, \dots, 0}_{m}, \underbrace{0, \dots, 0}_{\ell}, \underbrace{\rho}_{p}, \underbrace{1, 1, \dots, 1}_{n}, \underbrace{0, \dots, 0}_{q}, \underbrace{0, \dots, 0}_{n-1}]}^{N} \qquad (5.6a)$$

$$\rho_y = [\underbrace{0, \dots, 0}_{m}, \underbrace{1, \dots, 1}_{\ell}, \underbrace{\rho}_{p}, \underbrace{0, \dots, 0}_{q}, \underbrace{0, \dots, 0}_{n-1}, \underbrace{1, 1, \dots, 1}_{n}] \qquad (5.6b)$$

where $m \geq 0$ *and* $\ell \geq 0$ *are arbitrary,* $q = 0$, $\rho \in R^p$ *is an arbitrary (or null)* 0-1 *vector chosen identically in both* ρ_u *and* ρ_y; *and* $n > 0$ *is the order of the irreducible plant* (1?. 1). *Furthermore, if the system* (2. 1) *is obtained by a zero-order hold (ZOH) digitization of a continuous-time plant, the integer* $q \geq 0$ *can be chosen arbitrarily.*

Proof: The ZA conditions follow using the same arguments found in the proof of Theorem 1. Consider the IT condition (4.3). Given the! lifting (5.6), H_s has the form,

$$H_s = \begin{array}{c c} & \begin{array}{cc} p & n \end{array} \\ \begin{array}{c} p \\ n \\ \ell \end{array} & \begin{pmatrix} \mathcal{F} & \mathbf{0} \\ X & \mathcal{H} \\ X & X \end{pmatrix} \end{array} \tag{5.7}$$

From the structure of (5.7) it follows that H_s has full rank if both $\mathcal{F}$ and $\mathcal{H}$ have full rank. The remainder of the proof is identical to the proof of Theorem 1. ●

Properties of Lozano's $2n$ lifting follows directly from its interpretation as a special case of the liftings in Theorem 1 and Theorem 2.

Corollary 1 *Lozano's* $2n$*-lifting (3.12) satisfies the ZA, OT and IT conditions.*

Proof: The lifting (3.12) it is equivalent to the special case of the liftings in Theorems 1 and 2 where $m = 0$, $\ell = 0$, $p = 1$, $\rho = [0]$, $q = 0$. ■

G. DISCUSSION

The advantages of the extended horizon liftings in (5.1) and (5.6) relative to Lozano's $2n$-lifting are as follows,

(i) If one chooses $\ell > 0$ in (5.1), there are more control inputs than outputs in the lifted system (i.e., $\sigma_u > \sigma_y$). It is shown in Sect. 7 that these extra degrees of freedom can be used to design a controller which minimizes a quadratic control cost while simultaneously satisfying a deadbeat tracking objective. This significantly reduces control gains compared with Lozano's lifting.

(ii) If one chooses $\ell > 0$ in (5.6), there are more outputs than inputs in the lifted system (i.e., $\sigma_y > \sigma_u$). It is shown in Sect. 8 that these extra degrees of freedom can be used to minimize a quadratic error when estimating the input from measurements of the output. This is significant for reducing noise in problems of nonminimum phase channel equalization. Furthermore, the ρ vector can contain additional message information to increase the channel throughput.

(iii) If one chooses $m > 0$ in **(5.1)** or (5.2) there is an extra $m * T$ seconds of free time which can be used to perform computations (where T is the sampling interval). Since m can be chosen arbitrarily, the use of extended horizon liftings for either control or equalization applications is not constrained by real-time computer limitations. This is particularly useful for adaptive implementations which involve additional computation.

7. APPLICATION TO PLANT INVERSE CONTROL

The placement of the transmission zeros to the origin by the class of extended horizon liftings (5.1) allows stable invertibility of the transfer function from $U_s(k)$ to $Y_s(k)$. A control law which will be discussed next deadbeats the response $Y_s(k)$ to follow the desired $Y_d(k)$, subject to the minimization of a quadratic control cost.

To derive the desired controller, define the output error as,

$$E(k) = Y_d(k) - Y_s(k) \tag{7.1}$$

Substituting (4.5a) into (7.1) gives,

$$E(k) = -S_y A S_y^T Y_s(k-1) - H_s U_s(k) + Y_d(k) \tag{7.2}$$

Consider the problem of forcing the error in (5.3) to zero in a single step, while minimizing a quadratic control cost penalty, i.e.,

$$\min_{U_s(k)} \; U_s^T(k) U_s(k) \tag{7.3}$$

subject to

$$E(k) = 0 \tag{7.4}$$

In light of the OT condition (4.2), this minimization problem can be solved uniquely to give [5][6],

Zero Annihilation Periodic (ZAP) Control Law:

$$U_s^o(k) = H_s^\dagger\left(-S_y A S_y^T Y_s(k-1) + Y_d(k)\right) \tag{7.5}$$

$$= K^o Y_s(k-1) + L^o Y_d(k) \tag{7.6}$$

where the corresponding feedback gains are defined as,

$$K^o = -H_s^\dagger S_y A S_y^T \tag{7.7a}$$

$$L^o = H_s^\dagger \tag{7.7b}$$

Here the superscript "o" is chosen to emphasize the fact that the control nulls (i.e., deadbeats) the output. Also, in light of the OT condition, H_s has full row rank and one can write $H_s^\dagger = H_s^T(H_s H_s^T)^{-1}$ (cf., Barnett [2]).

For convenience the ZAP control law is summarized in the block diagram of Fig. 5. We have the following result.

Lemma 2 *(ZAP Control) Consider the closed-loop system arising from the OT extended horizon lifting (5.1) under ZAP control (7.6). Then,*

(i) *The quadratic control cost (7.3) is minimized at each stage, subject to the deadbeat tracking constraint (7.4),*

(ii) *All closed-loop poles are at the origin (i.e., the closed-loop response is deadbeat), and hence $Y_s(k)$ converges to $Y_d(k)$ in a single step,*

(iii) *The closed-loop system is internally stable (e.g., $Y_s^c(k)$ remains bounded).*

Proof: Result (i) follows from the OT condition (4.2) and well known minimum-norm properties of the Moore-Penrose inverse (cf., Barnett [2]). Now form the closed-loop system from the simplified lifted plant (4.5) under ZAP control (7.6),

$$\begin{bmatrix} Y_s(k) \\ Y_s^c(k) \\ U_s(k) \end{bmatrix} = A_{cl} \begin{bmatrix} Y_s(k-1) \\ Y_s^c(k-1) \\ U_s(k-1) \end{bmatrix} + B_{cl} Y_d(k) \qquad (7.8)$$

where,

$$A_{cl} = \begin{bmatrix} (I - H_s H_s^\dagger) S_y A S_y^T & 0 & 0 \\ S_y^c (I - H S_u^T H_s^\dagger S_y) A S_y^T & 0 & 0 \\ -H_s^\dagger S_y A S_y^T & 0 & 0 \end{bmatrix}; \quad B_{cl} = \begin{bmatrix} H_s H_s^\dagger \\ S_y^c H S_u^T H_s^\dagger \\ H_s^\dagger \end{bmatrix} \qquad (7.9)$$

Results (ii) and (iii) follow by substituting the OT condition (4.2) into (7.9), and noting that the resulting closed-loop matrix is stable with all of its eigenvalues at the origin. ●

Result (i) of Lemma 2 is important because it indicates that control gains associated with using extended horizon liftings will be significantly reduced compared to those from using the $2n$-lifting. Result (iii) of Lemma 2 is important because it ensures that the complementary output Y_s^c remains "well behaved" even though it is not being controlled directly.

Remark 1 Instead of deadbeat control, a pole placement scheme can be obtained by modifying the deadbeat constraint (7.4) to become $E(k) = \alpha E(k-1)$ in which case the ZAP control becomes $U_s^\circ(k) = K^\circ Y_s(k-1) + L^\circ Y_d(k) - \alpha L^\circ E(k-1)$. ■

Example 3 As an example, a 12-state nonminimum phase transfer function is shown in Fig. 6 (pole-zero plot, sampling time T = .025 secs), adopted from the ASTREX flexible structure model [6] [7]. The open-loop response to an initial condition is shown in Fig. 7. A simulation is first run using the deadbeat control (7.6) with the Lozano's $2n$-lifting, $m = 0$, $\ell = 0$, $p = 1$, $\rho = [0]$, $n = 12$, $q = 0$, $N = 2n = 24$, and using $Y_d = 0$ (i.e., a vibration damping objective). The response is shown in Fig. 8 to reach $\sim 3 \times 10^4$ at the output and 500 at the input. As expected from the theory, the response is deadbeat after a single horizon. However, *this control law is unusable since the units are in volts, and the allowable range is only ± 10 Volts.* It is emphasized that these extraordinarily large responses are typical of the $2n$-lifting due to the fact that the plant is inverted on a horizon of length $N \cdot T = .6$ seconds.

In an attempt to get a practical response with the $2n$-lifting, the pole placement control of Remark 1 is used. The choice $\alpha = .5$ is made to get approximately a 10 second decay time (any slower would be worse than the open-loop response). The results are simulated but not shown here since it turns out that the responses are reduced 50%, and are still unacceptably large by several orders of magnitude. The ZAP control using an extended horizon lifting $m = 0$, $\ell = 40$, $p = 0$, $q = 0$, $n = 12$, $N = 63$ is tried next. The results are shown in Fig. 9 where it is seem that both the input and output are well within the allowable ranges. The deadbeat nature of the response is also noted, as the vibrations are damped instantaneously after the first horizon. ■

8. APPLICATION TO CHANNEL EQUALIZATION

An important problem in communications is that of equalizing a nonminimum phase channel. The usual problem is that the channel cannot be inverted in a stable fashion. However, using the IT extended horizon lifting (5.6), this problem can be overcome.

Let $U_s(k)$ be the sequence of messages to be sent, and assume that an IT extended horizon lifting (5.6) is used to transmit the data (i.e., the signal sent is given by $U(k) = S_u^T U_s(k)$). The channel is assumed to be a stable linear nonminimum phase transfer function of the form (2.1) with order n. Then from Theorem 2 and Lemma 1, the plant dynamics are given by (4.5),

$$Y_s(k) = A_s Y_s(k-1) + H_s U_s(k) \tag{8.1}$$

where

$$A_s = S_y A S_y^T \tag{8.2}$$

At the receiving end, the quantity $Y_s(k) = S_y Y(k)$ is measured, and it is desired to estimate the messages $U_s(k)$ which were sent. For this purpose, an output prediction y.(k) is formed as,

$$\hat{Y}_s(k) = A_s Y_s(k-1) + H_s \hat{U}_s(k) \tag{8.3}$$

and an estimate $\hat{U}_s$ is found by minimizing the least squares criteria,

$$\min_{\hat{U}_s} (Y_s - \hat{Y}_s)^T (Y_s - \hat{Y}_s) \tag{8.4}$$

Since the IT condition (4.3) holds for the extended horizon lifting, the unique solution to (8.4) is given by,

$$\hat{U}_s = - H_s^\dagger (A_s Y_s(k-1) - Y_s(k)) \tag{8.5}$$

where one can write $H_s^\dagger = (H_s^T H_s)^{-1} H_s^T$ since H_s has full column rank (cf., Barnett [2]). It is noted that this dynamical system represents a stable inversion of the nonminimum phase channel characteristics. Furthermore, the use of an extended horizon $\ell > 0$ has allowed channel inversion by least squares which provides smoothing in case of noise. Lozano's $2n$-lifting in this application would not allow smoothing and would be very susceptible to noise. The use of p in the IT extended horizon lifting also provides a means for sending additional information with each packet, to improve the overall throughput.

9. CONCLUSIONS

A general class of liftings was defined and shown to have the same desirable zero annihilation properties of Lozano's $2n$-lifting. In contrast to the $2n$-lifting, all new liftings have horizons greater than $2n$, i.e., they are of the *extended horizon* type. The use of extended horizons resolves many difficulties associated with the $2n$-lifting. For example, a Zero Annihilation Periodic (ZAP) controller is defined for which the control gains can be significantly reduced relative to Lozano's lifting. This is due to a quadratic control cost which is minimized simultaneously with the deadbeat tracking objective. The effectiveness was shown in a simulation example where the control torque was reduced 4 orders of magnitude.

As a dual result, it was shown that a related class of liftings enables the on-line equalization of nonminimum phase channels in communication systems. This overcomes the standard bottleneck of inverting the channel in a stable fashion. Here, channel inversion is accomplished by least squares estimation which provides smoothing in the case of noise. It is worth noting that this channel equalization approach can be made adaptive by using standard recursive algorithms since the new liftings are linear-in-the-parameters. The full significance of this result remains to be explored.

It is expected that the results contained in this paper will be useful in many areas of modern control, neural control, fuzzy control, adaptive control, communications, adaptive filtering, signal processing, or other applications where a stable system inverse is desired but not possible due to nonminimum phase constraints.

ACKNOWLEDGEMENTS

This research was performed at the Jet Propulsion Laboratory, California Institute of Technology, under contract with the National Aeronautics and Space Administration.

REFERENCES

[1] P. Albertos, "Block multirate input-output model for sampled-data control systems," IEEE Trans. Automatic Control, vol. 35, no. 9, pp. 1085-1088, September 1990.

[2] S. Barnett, *Matrices: Methods and Applications.* Clarendon Press, Oxford England, 1990.

[3] D. S. Bayard, "Globally stable adaptive periodic control," Jet Propulsion Laboratory, Internal Document JPL D-9448, February 3, 1992.

[4] D. S. Bayard, "Zero annihilation methods for direct adaptive control of nonminimum phase systems," in *Proc. Seventh Yale Workshop on Adaptive and Learning Systems*, Yale University, May 1992.

[5] D.S. Bayard, "Extended horizon liftings," Jet Propulsion Laboratory, Internal Document, March 1993.

[6] D.S. Bayard and D. Boussalis, "Noncolocated structural vibration suppression using Zero Annihilation Periodic control," submitted to the 2nd IEEE Conference on Control Applications, Vancouver Canada, September 13-16, 1993.

[7] A. has, J. I,. Berg, G.A. Norris, D.F. Cossey, T.J. Strange III, and W.T. Schlaedgel, "ASTREX - A Unique test bed for CSI research," Proc. 29th IEEE Conference on Decision and Control, Honolulu, Hawaii, pp. 2018-2023, December 1990.

[8] E.J. Davison and S.H. Wang, "Properties and calculation of transmission zeros of linear multivariable systems," Automatica, Vol. 10, pp. 643-658, 1974.

[9] R. Lozano-Leal, "Robust adaptive regulation without persistent excitation," IEEE Trans. Automatic Control, vol. 34, no. 2, pp. 1260-1267, December 1989.

[10] R. Lozano, J-M. Dion, and L. Dugard, "Sillglllality-flee adaptive pole placement using periodic controllers," IEEE Trans. Automatic Control, vol. 38, no. 1, pp. 104-108, January 1993.

[11] R. Ortega, "On periodic adaptive stabilization of non-minimum phase systems," Proc. 30th IEEE Conf. on Decision and Control, Brighton, England, December 1991.

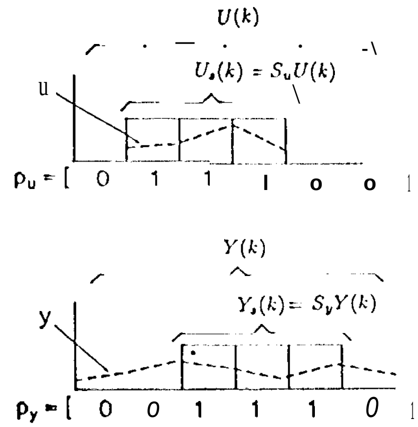

Figure 1: Vectors U_s and Y_s defined using windows ρ_u and ρ_y, respectively.

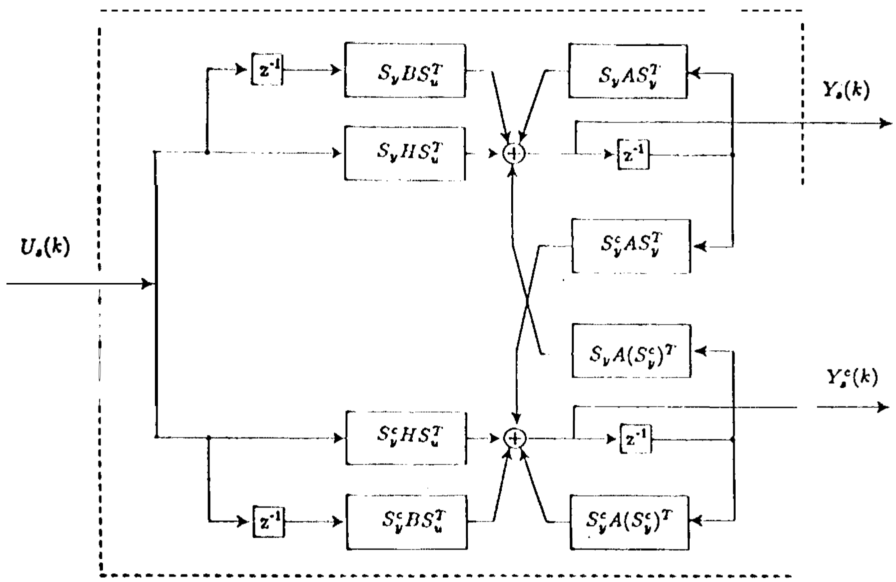

Figure 2: Generalized Lifting System Model, $\mathcal{G}(\rho_u, \rho_y)$.

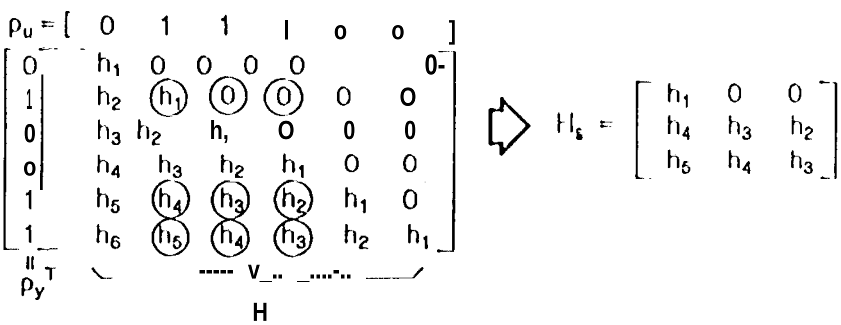

Figure 3: Determination of H_s from input and output windows ρ_u and ρ_y.

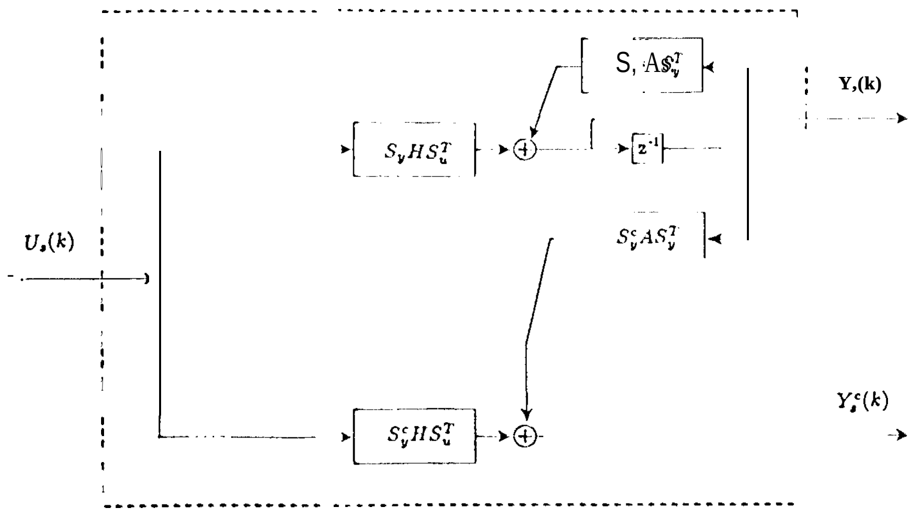

Figure 4: Generalized Lifting System Model under the Zero Annihilation (ZA) conditions $BS_u^T = 0$ and $A(S_y^c)^T = 0$.

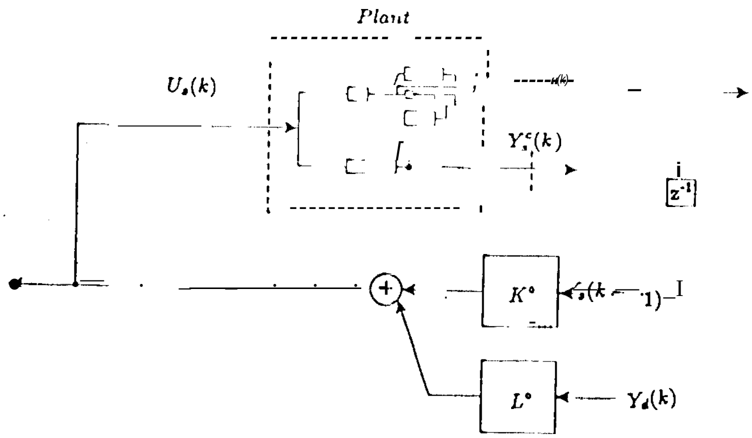

Figure 5: Zero Annihilation Periodic (ZAP) control law architecture.

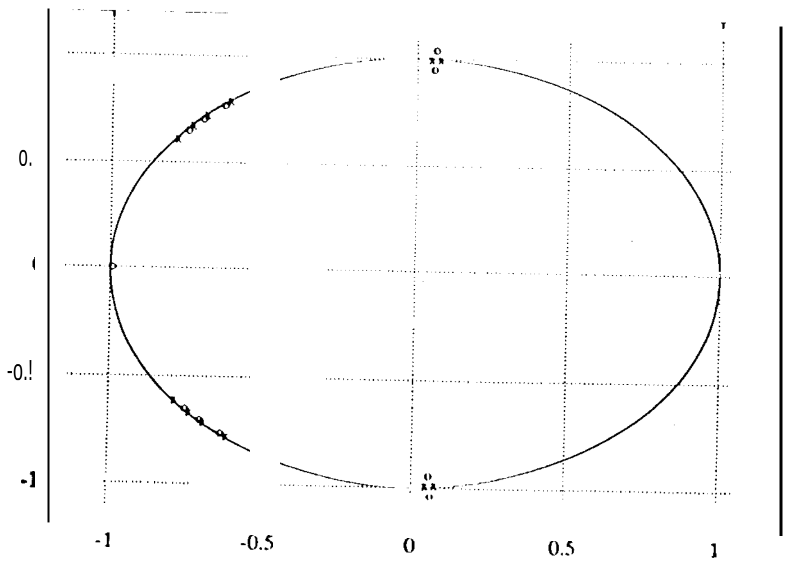

Figure 6: Pole-zero plot of nonminimum phase plant, $T = .025$ sec sampling interval: poles '(x", zeros 'o"

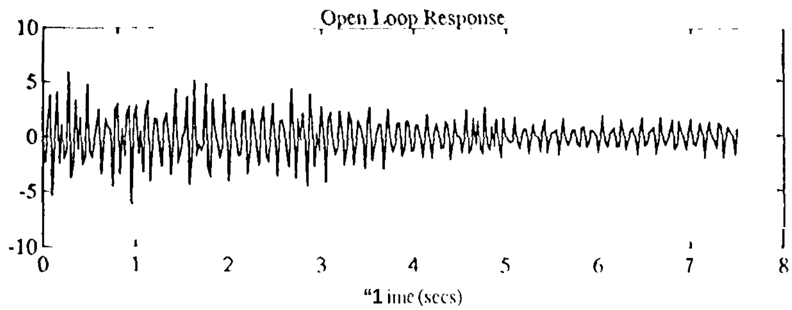

Figure 7: Open-loop response (plant output).

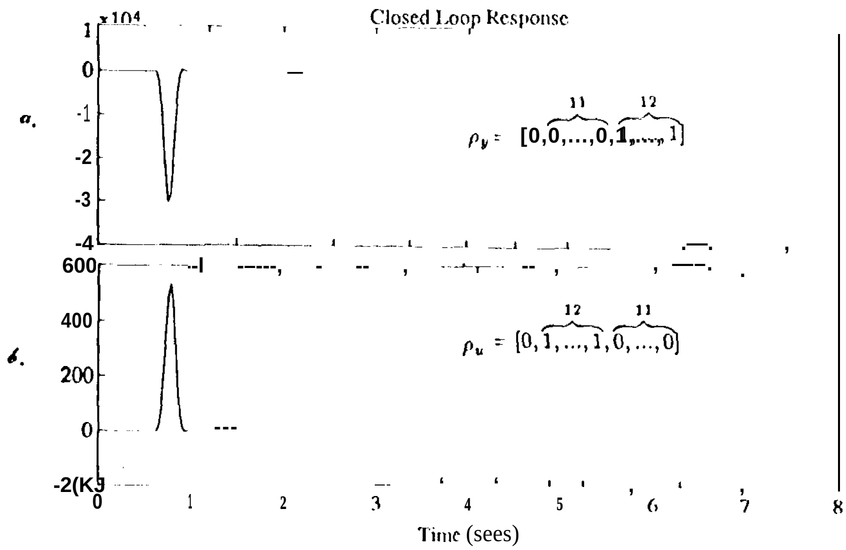

Figure 8: Closed-loop response using deadbeat control with Lozano lifting: a. plant output; b. control input.

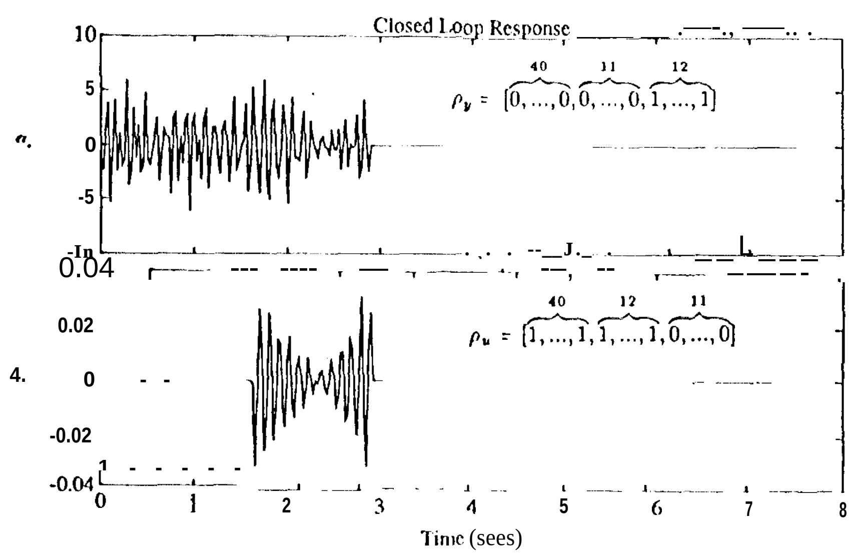

Figure 9: Closed-loop response using ZAP control with extended horizon lifting $m = 0$, $\ell = 40$, $p = 0$, $q = 0$, $n = 12$, $N = 64$: a. plant output; b. control input.