Computation of H_∞ Norm for Flexible Structures

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Abstract. *The H_∞ norm for a system with flexible structural properties is a function of its largest Hankel singular value.*

The computation of the H_∞ norm of a linear system is a computationally intensive search process [1-3], specifically in the case of flexible systems. This paper presents a direct way to determine the H_∞ norm for flexible structures which avoids iterations.

In this paper a flexible structure is defined as a controllable and observable linear system with distinct complex conjugate pairs of poles (N poles, N is even), and with small negative real parts of the poles. In the Moore balanced coordinates [4] it consists of $n = N/2$ components [5, 6], and each component consists of two states. Let (A, B, C) be a state-space triple of a flexible structure, and $G = C(sI-A)^{-1}B$ its transfer function, with the H_∞ norm defined as follows

$$\|G\|_\infty = \sup_\omega \sigma_{max}(G(j\omega)) \tag{1}$$

The system controllability and observability grammians W_c and W_o are positive-definite and satisfy the Lyapunov equations

$$AW_c + W_cA^T + BB^T = 0, \qquad A^TW_o + W_oA + C^TC = 0. \tag{2}$$

The system representation is balanced in the sense of Moore (cf. [4]) if its controllability and observability grammians are equal and diagonal

$$W_c = W_o = \Gamma^2, \quad \Gamma = diag(\gamma_1, \ldots, \gamma_N) \tag{3}$$

where $\gamma_i > 0$ is the i-th Hankel singular value of the system, and the Hankel singular values are in the decreasing order, $\gamma_i \geq \gamma_{i+1}$, $i = 1, \ldots, N-1$.

Further, an approximate equality between two variables is used in the following sense. Two variables x and y are approximately equal ($x \cong y$) if $x = y + \varepsilon$, and $\|\varepsilon\|/\|y\| \ll 1$. It is shown, see [5, 6], that for a balanced flexible system (A, B, C) with n components (or $N = 2n$ states), the balanced grammian has the following form

$$\Gamma \cong diag(\gamma_1, \gamma_1, \gamma_2, \gamma_2, \ldots, \gamma_n, \gamma_n) \tag{4}$$

and the matrix A is almost block-diagonal, with dominant 2x2 blocks on the main diagonal

$$A \cong diag(A_i), \quad A_i = \begin{bmatrix} -\zeta_i\omega_i & -\omega_i \\ \omega_i & -\zeta_i\omega_i \end{bmatrix} \quad i = 1, \ldots, n \tag{5}$$

where ω_i is the i-th natural frequency of the structure, and ζ_i is the ith modal damping. Note also that introducing (4) and (5) to (2) gives

$$\gamma_i^2(A_i + A_i^T) \cong B_iB_i^T \cong -C_i^TC_i \tag{6}$$

and B_i, C_i are i-th row and column of B and C, respectively.

The H_∞ norm is evaluated from the Riccati equation, as shown in [1-3]. It is the smallest positive parameter ρ such that the solution of the following Riccati equation is positive-definite

$$A^TS + SA + \rho^{-2}SBB^TS + C^TC = 0. \tag{7}$$

From this definition it follows that:

Proposition. For G being a transfer function of a flexible structure in the state space representation (A, B, C), its H_∞ norm is as follows

$$\|G\|_\infty \cong 2\gamma_1^2 \tag{8}$$

where γ_1 is the largest Hankel singular value of the system.

Proof. For a flexible structure, due to properties (4)(6) the solution S of the Riccati equation (7) is diagonally dominant, see [7]

$$S \cong diag(s_1, s_1, \ldots, s_n, s_n) \tag{9}$$

where s_i is a solution of the following equation

$$s_i(A_i + A_i^T) + s_i^2\rho_i^{-2}B_iB_i^T + C_i^TC_i \cong 0, \quad i = 1, \ldots, N \tag{10}$$

and A_i, B_i, and C_i are given in (5). Introducing (6) to (10) one obtains

$$s_i^2(A_i + A_i^T) - s_i\gamma_i^2\rho_i^2(A_i + A_i^T) - \gamma_i^2(A_i + A_i^T) \cong 0 \tag{11}$$

or

$$s_i^2 - s_i\rho_i^2\gamma_i^{-2} + \rho_i^2 \cong 0 \tag{12}$$

with two solutions $s_i^{(1)}$ and $s_i^{(2)}$

$$s_i^{(1)} \cong 0.5\rho^2\gamma^{-2}(1-\beta_i), \quad s_i^{(2)} \cong 0.5\rho^2\gamma^{-2}(1+\beta_i), \tag{13a}$$

$$\beta_i = \sqrt{1 - 4\gamma_i^4/\rho_i^2} \tag{13b}$$

For $\rho_i = 2\gamma_i^2$ one obtains $s_i^{(1)} = s_i^{(2)} = 2$, and $\rho_i = 2\gamma_i^2$ is the smallest value of ρ, for which a positive solution s_i exists. It is indicated by the plots of $s_i^{(1)}$ and $s_i^{(2)}$ vs ρ, in Fig. 1. To obtain S positive definite, all s_i must be positive. Thus the the largest ρ from the set $\{\rho_1, \rho_2, \ldots, \rho_n\}$ is the smallest one for which S>0, therefore

$$\|G\|_\infty = \max_i \rho_i \cong 2\gamma_1^2 \qquad \text{o} \tag{14}$$

The proposition shows that the computation of the H_∞ norm consists of a standard procedure of computing the system Hankel singular values, and that the largest Hankel singular value determines the required norm.

Example. A truss structure as in Fig.2 is investigated. For this structure $l_1 = 70$ in., $l_2 = 100$ in.; each truss has a cross-section area of 2 in.^{2}, elastic modulus of 10^6 lb/in.^{2}, and mass density of 2 lb sec^2/in^2. Vertical control forces are applied at nodes *na1* and *na2*, and the output rates are measured in the vertical direction at nodes *no1* and *no2*. The system has 26 states (13 balanced components), two inputs, and two outputs. Its H_∞ norm is computed through iterations of Riccati equation (7), obtaining $\|G\| = 1.340852$. It took 24 iterations to obtain the required accuracy $\varepsilon = 10^6$, as shown in Fig.3. The H_∞ norm is also obtained from (8), and in this case $\|G\|_\infty = 1,340850$, with the same accuracy $\varepsilon = 10^6$.

In conclusion, the estimate of the H_∞ norm (8) is obtained without costly searching, with the estimation error close to the machine zero.

Acknowledgment. This research was performed at the Jet Propulsion Laboratory, California Institute of Technology, under a contract with the National Aeronautics and Space Administration.

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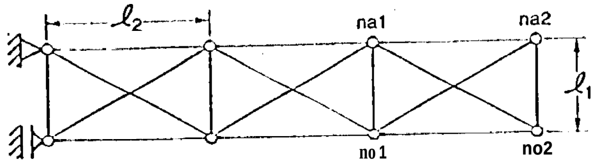

Fig.2. Truss structure

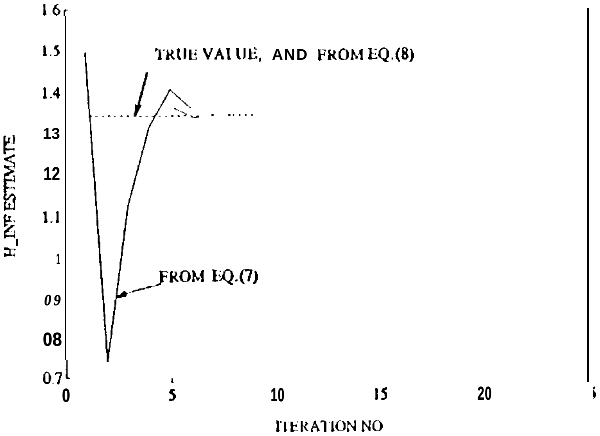

Fig.3. H_∞ norm of truss structure

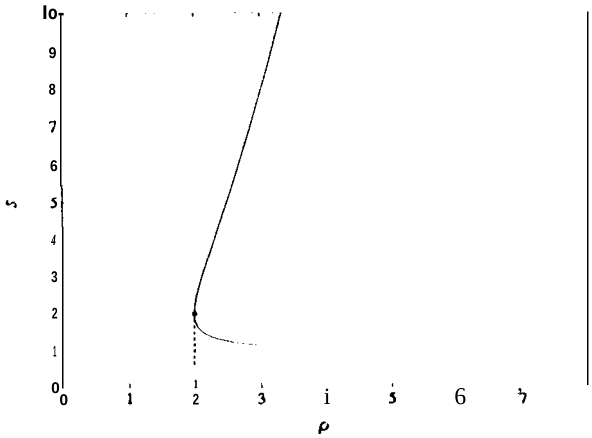

Fig.1. Solution of Riccati equation