Jacobi's Integral and ΛV-Earth- Gravity-Assi st (ΛV-EGA) Trajectories*

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by

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One of the most powerful tools in the trajectory designer's bag of tricks is the ΛV-Earth-Gravity-Assist (ΛV-EGA). On this trajectory a spacecraft l caves Earth on, for exampl o, a two-year heli ocentri c orbit. At the apheli on of that orbit a ΛV maneuver is performed which reshapes the orbit to lower its perihelion. The orbit timing is arranged so that when the spacecraft then crosses Earth's orbit (either before or after perihelion) it encounters Earth in a gravity assist maneuver. The advantage of the ΛV-EGA is that the increase in perigee velocity from the launch to the en counter is much greater than the vel ocity change at the d eep-space man euver. The resulting heliocentric energy is greater than in the initial two-year orbit and again the increase is more than could be obtained from the deep-space maneuver alone.

Two aspects of the ΛV-EGA arc counterintuitive (as is often the case in orbital mechanics). One is that the deep-space maneuver which sets up the heliocentric energy gain actuall y reduces the heliocentric energy--the spacecrafts] ows down at apheli on to move the periheli on c1 oscr to the sun. The second aspect is perhaps more puzzling to the experienced trajectory designer (who is used to the sometimes paradoxical behavior of orbits). In general, the most efficient time to change orbital energy is when the velocity is highest, i.e., at periapse. in the ΛV-EGA, however, the deep-space maneuver is most effective when done where the velocity is lowest, i.e., at aphelion.

The conventional explanation of the ΛV-EGA ignores these aspects as follows: it is easiest to reshape an orbit where the velocity is lowest; the more the orbit shape is changed, the greater the angl c between the spacecraft orbit and Earth's orbit where they cross; the greater the angle, the greater the di fference between the spacecraft's velocity and Earth's velocity at encounter; this velocity difference when aligned with

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Earth's velocity by the, gravity assist gives us our final heliocentric energy. All this is true and is fine as far as it goes. Nevertheless, there remains an element of mystery in the ΔV-EGA, not, least because there is no direct way in the conventional explanation to relate the magnitude of the deep-space maneuver to the final gain in heliocentric energy.

Jacobi's integral saves the day

The reason for the apparent mysteriousness of the ΔV-EGA trajectory is that the discussion above considers the trajectory as a series of two-body problems: Earth/spacecraft for launch, Sun/spacecraft for initial orbit and deep-space maneuver, Earth/spacecraft for gravity assist maneuver, and Sun/spacecraft for final orbits. But the ΔV-EGA is very much a creature of the three-body problem, in which it is not appropriate to base an analysis on energy. Instead, we must turn to the three-body analog of energy, Jacobi's integral.

If the Earth traveled in a circular orbit around the Sun and the only accelerations experienced by a (massless) spacecraft were caused by the central gravity of the Earth and Sun, then Jacobi's integral [1,2]

$$C = -v^2 + \omega^2 \rho^2 + \frac{2\mu_e}{r_e} + \frac{2\mu_s}{r_s} \qquad (1)$$

is a constant along the spacecraft's trajectory, where v is the magnitude of the **rotational velocity,** which is the velocity of the spacecraft in a rotating three dimensional coordinate system that is centered at the Earth-Sun barycenter and rotates with the Earth-Sun system, p is the distance from the barycenter to the projection of the spacecraft's position on the Earth-Sun orbit plane, r_e is the distance of the spacecraft from the Earth, and r_s is the distance of the spacecraft from the Sun. (Definitions for ω, μ_e, and μ_s are given in Table 1.)

In a two-body problem, energy is a constant function of position and the magnitude of the inertial velocity. in the circular restricted three-body problem, Jacobi's integral is a constant function of position and the magnitude of the rotational velocity. For our purposes here we may consider a maneuver to be an instantaneous velocity change which does not affect position. Thus, while an energy change is maximized for a maneuver if the maneuver is done when the inertial velocity is greatest (at the periapse of a conic), a change in Jacobi's constant is maximized if a maneuver is done when the rotational velocity is greatest.

This is the key to understanding the ΔV-EGA. The deep-space maneuver is in fact done when the magnitude of the rotational velocity is greatest and is done in the direction of the rotational velocity. Furthermore, the consequent change in Jacobi's constant can bc used to estimate the velocity increase from launch perigee to encounter perigee which results from the dcwp-space maneuver, so the magnifying effect of the ΛV-EGA can bc calcul atcd.

A numerical example

Of course the real world is not a circular restricted three-body problem. Nor has a straightforward ΛV-EGA trajectory been flown in a space mission. But ΛV-EGA trajectories have been carried as baseline trajectories during the design process of several missions. In particular, at one time the baseline trajectory for CRAF[3] used a two-year ΛV-EGA which had a deep-space maneuver of 0.6 km/s and an increase in perigee velocity of **2.2** km/s (see Figure 1). l ret's compare this to an estimate obtai ned by using J acobi's con stant.

Wc start by assuming the Earth travels in a circular orbit around the Sun according to the constants in 'l'able 1. A two-year orbit which is tangent at perihelion to Earth's orbit has an aphelion distance of 2.175 AU. At that distance, a point fixed in the rotating Earth-Sun system has an inertial velocity of 64.78 km/s (= 2.175 $a\omega$) in the direction of the rotation; equivalently a point fixed in inertia] space at that distance has a rotational velocity of equal magnitude but in the opposite direction. Since the spacecraft speed at aphelion is 16.03 km/s, the spacecraft's rotational velocity there is 48.75 km/s.

A point fixed close to Earth, say at 170 km altitude, has a negligible rotational vc1 ocity in the Earth-Sun rotating system. Thus for the launch ant] encounter, the spacecraft's rotati onal vel ocity is essentiall y th c same as its Narth-relative vc1 oci ty regardless of the orientation of the hyperbola. This is 12,15 km/s at 170 km altitude on a h yperbol a launchi ng tangentially into a two-year heli ocentri c orbit.

From equation (1) wc have

$$--AC = 2vAv + (\Lambda v)^2 \qquad (2)$$

so that for small *Av wc scc* that *ΛC is* roughly proportional to the rotational velocity. For the case analyzed here this gives about a magnification factor of 4, in good agreement with the data. More preci sel y, if v_a is the rotational velocity at aphelion and v_E is the rotational velocity at perigee, wc have

$$2\ v_E\ A\ v_E\ -\!\!\imath (\Lambda v_E)^2 =\!-\ 2\ v_a\ A\ v_a\ -i\ (\Lambda v_a)^2 \qquad (3)$$

or

$$(\Lambda v_E)^2 + 2912.\ 15\Lambda v_E\ -2\text{-}48.\ 75\cdot 0.6 - 0.6^2 = O \qquad (4)$$

so that Λv_E = 2.,19 km/s, in even better agreement with the data.

A new type of ΔV-EGA trajectory

The example above was a two-year ΔV-EGA, but of course there is nothing to constrain the! initial orbit to have a two-year period, A three-year orbit would do as well and in fact, as this analysis implies, gives a greater magnification of the deep-space maneuver. A less commonly considered alternative is a 1.5-year orbit which encounters Earth after three years.

All that is really necessary for a ΔV-EGA trajectory is that a spacecraft leave Earth on an orbit to a point where its rotational velocity is greater than at launch perigee and from whore it can encounter Earth after performing a maneuver.

This leads us to realize the existence of a new type of ΔV-EGA trajectory. The ΔV-EGAs above all start with orbits larger than Earth's orbit; let's call them external ΔV-EGA trajectories. What about internal ΔV-EGA trajectories, that start off with orbits small er than Earth's?

For example, if a spacecraft starts off in a 2/3- year orbit it leaves with very nearly the same velocity relative to Earth as in a two-year orbit but in the opposite direction. At perihelion the spacecraft has a rotational velocity of 31.32 km/s so a magnification factor of about 2 1/2 is possible for this ΔV-EGA. Internal ΔV-EGA trajectories have potential application to inner planet mi ssions and for reducing rendezvous ΔV in Mars or outer planet missions.

References

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Table 1.

CONSTANTS

a = 149597870. km	mean semi-major axis of the Earth's orbit
μ_s = 1.327124 x 10^{11} km^3/s^2	Gravitational constant times the mass of the Sun
μ_e = 398600.5 km^3/s^2	Gravitational constant times the mass of the Earth
ω = 1.990987 7 x 10^{-7} rad/s	mean angular rotation rate of the Earth-Sun system

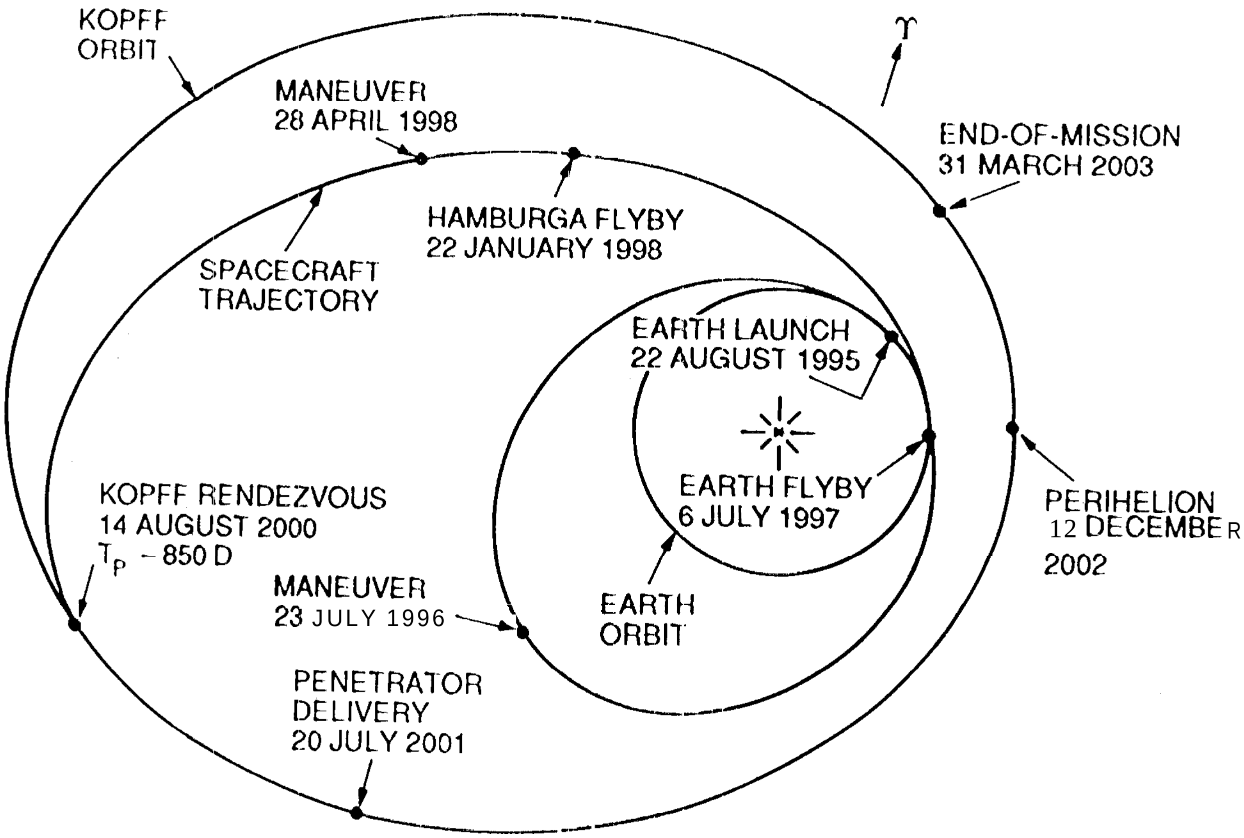

Figure 1. The two-year ΔV-EGA trajectory which was the baseline for CRAF in 1989.