Jacobi's Integral and AV-Earth-Gravity-Assist (AV-EGA) Trajectories*

An Abstract for Consideration for the 1993 AAS/AIAA Astrodynamics Conference
by
Theodore J. Sweetser, Ph.D.*

Jet \'repulsion Laboratory
California Institute of Technology
Pasadena, California

One of the most powerful tools in the trajectory designer's bag of tricks is the AV-Earth-Gravity-Assist (AV-EGA). On this trajectory a spacecraft caves Earth on, for example, a two-year heliocentric orbit. At the aphelion of that orbit a AV maneuver is performed which reshapes the orbit to lower its perihelion. The orbit timing is arranged so that when the spacecraft then crosses Earth's orbit (either before or after perihelion) it encounters Earth in a gravity assist maneuver. The advantage of the AV-EGA is that the increase in perigee velocity from the launch to the encounter is much greater than the velocity change at the deep-space maneuver. The resulting heliocentric energy is greater than in the initial two-year orbit and again the increase is more than could be obtained from the deep-space maneuver alone.

Two aspects of the AV-EGA are counterintuitive (as is often the case in orbital mechanics). One is that the deep-space maneuver which sets up the heliocentric energy gain actually reduces the heliocentric energy—the spacecraft's owns down at aphelion to move the perihelion closer to the sun. The second aspect is perhaps more puzzling to the experienced trajectory designer (who is used to the sometimes paradoxical behavior of orbits). In general, the most efficient time to change orbital energy is when the velocity is highest, i.e., at perigee. In the AV-EGA, however, the deep-space maneuver is most effective when done where the velocity is lowest, i.e., at aphelion.

The conventional explanation of the AV-EGA ignores these aspects as follows: it is easiest to reshape an orbit where the velocity is lowest; the more the orbit shape is changed, the greater the angle between the spacecraft orbit and Earth's orbit where they cross; the greater the angle, the greater the difference between the spacecraft's velocity and Earth's velocity at encounter; this velocity difference when aligned with

---

* Mathematician, Member AAS, AIAA

* The research described in this abstract was performed at the Jet \'repulsion Laboratory, California Institute of Technology, under contract with the National Aeronautics and Space Administration.
Earth's velocity by the gravity assist gives us our final heliocentric energy. All this is true and is fine as far as it goes. Nevertheless, there remains an element of mystery in the AV-EVA, not least because there is no direct way in the conventional explanation to relate the magnitude of the deep-space maneuver to the final gain in heliocentric energy.

Jacobi’s integral saves the day

The reason for the apparent mysteriousness of the AV-EVA trajectory is that the discussion above considers the trajectory as a series of two-body problems: Earth/spacraft for launch, Sun/spacraft for initial orbit and deep-space maneuver, Earth/spacraft for gravity assist maneuver, and Sun/spacraft for final orbits. But the AV-EVA is very much a creature of the three-body problem, in which it is not appropriate to base an analysis on energy. Instead, we must turn to the three-body analog of energy, Jacobi’s integral.

If the Earth traveled in a circular orbit around the Sun and the only accelerations experienced by a (massless) spacecraft were caused by the central gravity of the Earth and Sun, then Jacobi’s integral is

$$C = -\frac{2}{r_c} - \frac{2}{r_s} + 2\mu_e \cdot \frac{\omega}{r_c} + 2\mu_s \cdot \frac{\omega}{r_s}$$

is a constant along the spacecraft’s trajectory, where \(v\) is the magnitude of the rotational velocity, which is the velocity of the spacecraft in a rotating three dimensional coordinate system that is centered at the Earth-Sun barycenter and rotates with the Earth-Sun system, \(p\) is the distance from the barycenter to the projection of the spacecraft’s position on the Earth-Sun orbit plane, \(r_c\) is the distance of the spacecraft from the Earth, and \(r_s\) is the distance of the spacecraft from the Sun. (Definitions for \(\omega\), \(\mu_e\), and \(\mu_s\) are given in Table 1.)

In a two-body problem, energy is a constant function of position and the magnitude of the inertial velocity. In the circular restricted three-body problem, Jacobi’s integral is a constant function of position and the magnitude of the rotational velocity. For our purposes here we may consider a maneuver to be an instantaneous velocity change which does not affect position. Thus, while an energy change is maximized for a maneuver if the maneuver is done when the inertial velocity is greatest (at the periapse of a conic), a change in Jacobi’s constant is maximized if a maneuver is done when the rotational velocity is greatest.

This is the key to understanding the AV-EVA. The deep-space maneuver is in fact done when the magnitude of the rotational velocity is greatest and is done in the direction of the rotational velocity. Furthermore, the consequent change in Jacobi’s
constant can be used to estimate the velocity increase from launch perigee to encounter perigee which results from the dcwp-space maneuver, so the magnifying effect of the ΔV-EQA can be calculated.

A numerical example

Of course the real world is not a circular restricted three-body problem. Nor has a straightforward ΔV-EQA trajectory been flown in a space mission. But ΔV-EQA trajectories have been carried as baseline trajectories during the design process of several missions. In particular, at one time the baseline trajectory for CRAIR3 used a two-year ΔV-EQA which had a deep-space maneuver of 0.6 km/s and an increase in perigee velocity of 2.2 km/s (see Figure 1). Let's compare this to an estimate obtained by using Jacobi's constant.

We start by assuming the Earth travels in a circular orbit around the Sun according to the constants in Table 1. A two-year orbit which is tangent at perihelion to Earth's orbit has an aphelion distance of 2.175 AU. At that distance, a point fixed in the rotating Earth-Sun system has an inertial velocity of 64.78 km/s (= 2.175 a o) in the direction of the rotation; equivalently a point fixed in inertial space at that distance has a rotational velocity of equal magnitude but in the opposite direction. Since the spacecraft speed at aphelion is 16.03 km/s, the spacecraft's rotational velocity there is 48.75 km/s.

A point fixed close to Earth, say at 170 km altitude, has a negligible rotational velocity in the Earth-Sun rotating system. Thus for the launch and encounter, the spacecraft's rotational velocity is essentially the same as its North-relative velocity regardless of the orientation of the hyperbola. This is 12.15 km/s at 170 km altitude on a hyperbola launching tangentially into a two-year heliocentric orbit.

From equation (1) we have

\[ \Delta V = 2v \Delta v + (\Delta v)^2 \]  

so that for small \( \Delta v \) we see that \( \Delta V \) is roughly proportional to the rotational velocity. For the case analyzed here this gives about a magnification factor of 4, in good agreement with the data. More precisely, if \( v \) is the rotational velocity at aphelion and \( v_f \) is the rotational velocity at perigee, we have

\[ 2v_f: A v_f: \cdot (\Delta v_f)^2 = 2 v_f A v \cdot (\Delta v_\alpha)^2 \]  

or

\[ (\Delta v_f)^2 + 2912.15v_f - 2.48.75 \cdot 0.6 - 0.6^2 = 0 \]  

so that \( \Delta v_f = 2.19 \) km/s, in even better agreement with the data.
A new type of AV-EGA trajectory

The example above was a two-year AV-EGA, but of course there is nothing to constrain the initial orbit to have a two-year period. A three-year orbit would do as well and in fact, as this analysis implies, gives a greater magnification of the deep-space maneuver. A less commonly considered alternative is a 1.5-year orbit which encounters Earth after three years.

All that is really necessary for a AV-EGA trajectory is that a spacecraft leave Earth on an orbit to a point where its rotational velocity is greater than at launch perigee and from where it can encounter Earth after performing a maneuver.

This leads us to realize the existence of a new type of AV-EGA trajectory. The AV-EGAs above all start with orbits larger than Earth's orbit; let's call them external AV-EGA trajectories. What about internal AV-EGA trajectories, that start off with orbits smaller than Earth's?

For example, if a spacecraft starts off in a 2/3-year orbit it leaves with very nearly the same velocity relative to Earth as in a two-year orbit but in the opposite direction. At perihelion the spacecraft has a rotational velocity of 31.32 km/s so a magnification factor of about 2 1/2 is possible for this AV-EGA. Internal AV-EGA trajectories have potential application to inner planet missions and for reducing rendezvous AV in Mars or outer planet missions.

References


Table 1.

CONSTANTS

\[ a = 149597870 \text{ km} \]
\[ \mu_S = 1.327124 \times 10^{11} \text{ km}^3/\text{s}^2 \]  
Gravitational constant times the mass of the Sun
\[ \mu_C = 398600.5 \text{ km}^3/\text{s}^2 \]  
Gravitational constant times the mass of the Earth
\[ \omega = 1.990987 \times 10^{-7} \text{ rad/s} \]  
Mean angular rotation rate of the Earth-Sun system

Figure 1. The two-year AV-EGA trajectory which was the baseline for CRAF in 1989.