Linear Algorithms for Worst Case Identification in $\mathcal{H}^\infty$ with Applications to Flexible Structures

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Abstract

This paper is concerned with linear algorithms for identification in $\mathcal{H}^\infty$ which have been studied in [9]. It is shown that the two different linear algorithms in [9] can be unified into a single one which can be further extended to nonuniformly spaced frequency response samples with exponential convergence for the noise free case. Improved upper bounds for the corresponding identification errors arc derived. Applications to the identification of lightly damped systems such as flexible structures arc also considered.

1 Introduction

Recently, a control oriented identification problem has been formulated by Helmicki, Jacobson and Nett [6]: given a finite number of noisy experimental frequency response data, find an algorithm which not only identifies the nominal plant model, but also quantifies the worst case identification error in $\mathcal{H}^\infty$ norm. Further, the algorithm is required to have the property that ihc worst case identification error converge to zero as the noise level goes to zero and the number of experimental data points goes to infinity. This particular identification problem is termed as identification in $\mathcal{H}^\infty$, and is mainly motivated by the need of modern robust control theory. in the context of feedback system design, it is essential that the resulting system identification algorithm produce an identified mode] that converges in a topology for which feedback stability is a robust proper-(y. Sue]) topology is chosen as $\mathcal{H}^\infty$ that is consistent with the robust control design. The research work along this direction constitutes an important part of robust identification. See []]- [5], [7]-[1 4], [] 6]-[19] and references cited therein.

in this paper, linear algorithms for identification in $\mathcal{H}^\infty$ will be considered due to their simplicity and efficiency. An algorithm is said to be *tuned* if the *a priori* information of the plant model or/and noise level is used in the identification process. Since *untuned* linear algorithms are divergent in face of the worst case noise [] 4], convergent linear algorithms arc necessarily *tuned*. A class of tuned linear algorithms are thmc reported in [9] based on the least square fitting. While it remains unknown for the existence of other types of tuned linear algorithms which arc convergent, our study reveals some interesting features of the least square based linear algorithms [9]. The most important onc is that the linear algorithms developed in [9] are exponentially convergent for nonuniformly spaced frequency response data. Further, the two seemingly different linear algorithms in [9] can in fact be unified into a single onc which arc applicable to the problem of identification in $\mathcal{H}^\infty$ for both uniformly and nonuniformly spaced frequency response data. Improved upper bounds arc derived for the least square based linear algorithm in [9]. It is also interesting to note that the linear algorithms studied in this paper, combined with the balanced model reduction, give an effective procedure for the identification of lightly damped systems.

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2 Problem Formulation and Preliminaries

Let $\mathcal{T}$ denot c the unit circle of the complex plane C and $\mathcal{H}^\infty$ denote the collection of all functions which arc essentially bounded in T and which admit analytic extension on the open unit disc 'D. An open open disc of radius p > 0 will be denoted by $\mathcal{D}_\rho$. Under the norm

$$\|\hat f\|_\infty = \sup_{z\in\mathcal{D}} |\hat f(z)|, \quad \hat f \in \mathcal{H}^\infty,$$

$\mathcal{H}^\infty$ is a Banach space. The class of systems under consideration consists of stable, causal, linear shift invariant discrete-time systems that admit transfer function

$$\hat h(z) = \sum_{k=0}^{\infty} h_k z^k \in \mathcal{H}^\infty,$$

with impulse response $\{h_k\}$. The $\mathcal{Z}$-transform is so defined that the stability corresponds to no pole on the closed unit disc. The true, unknown system to be identified is assumed to be in the set

$$\mathcal{H}(M,\ \rho) = \left\{\hat h : \hat h \in \mathcal{H}^\infty \text{ and } |\hat h(z)| \le M,\ \forall z \in \mathcal{D}_\rho\right\} \tag{1}$$

where M > 0, $\rho > 1$. The value of M represents the system gain over all exponentially weighted sinusoidal inputs while the value of p represents the relative stability of the system]. The pair (M, ρ) characterizes the *a priori* information of the system to be identified which can be experimentally estimated. The experimental data consists of a finite collection of frequency response samples corrupted by noise η given by

$$E_k^N = \hat h(e^{j\omega_k}) + \eta_k,\ \omega_k \in [-\pi,\ \pi],\ |\eta_k| \le \epsilon, \tag{2}$$

where k = 1 ,..., N and ϵ >0. Since a physical system has real impulse response, its frequency response satisfies the property of conjugate symmetry. It is thus clear that wc need perform frequency response experimentally for only positive frequencies. The problem of identification in $\mathcal{H}^\infty$ is to find an identification algorithm AN which maps the experimental data (2), and *a* priori information (M,ρ,ϵ) 10 an identified model $\hat h_{id} \in \mathcal{H}^\infty$ of degree n, and to an identification error $e_n(A_N, M, p, \epsilon)$ defined by

$$e_n(A_N, M, \rho, \epsilon) := \sup_{\hat h\in\mathcal{H}(M,\rho),\|\eta\|_{\ell^\infty}\le\epsilon} \|\hat h_{id} - \hat h\|_\infty \tag{3}$$

Further, the algorithm A_N is required to be convergent in the sense that

$$\lim_{\epsilon\to 0, N, n\to\infty} e_n\ (AN,\ M, \rho, \epsilon) = 0.$$

The algorithms interest to us in this paper are the *tuned* linear algorithms reported in [9]. As mentioned earlier, such linear algorithms have some interesting features whit]) are not clear in [9]. The purpose of this paper is to give a unified treatment of least square based linear algorithms. The following lemma is important in clarifying the two different linear algorithms in [9] which is referred mixed Parserval theorem.

Lemma 2.1 *Let $\hat{p}(e^{j\omega}) = p_0 + p_1 e^{j\omega} + \ldots + p_{n-1} e^{j(n-1)\omega}$ Denote $W_N = e^{j2\pi/N}$. Then, for any $N \geq n$,*

$$\sum_{i=0}^{n-1} |p_i|^2 = \frac{1}{2\pi}\int_0^{2\pi} |\hat{p}(e^{j\omega})|^2\, d\omega = \frac{1}{N}\sum_{k=0}^{N-1} |\hat{p}(W_N^k)|^2.$$

Since the proof of the mixed Parserval theorem is elementary, it is omitted. Another result which is useful to provide improved error bounds in the next section is on the quantification of the $\mathcal{H}^\infty$ *norm* in terms of the $\mathcal{H}^2$ norm.

Lemma 2.2 *Denote $\mathcal{H}^2(\mathcal{D}_\rho)$ as the collection of functions analytic on $\mathcal{D}_\rho$, and absolutely square integrable on the boundary of $\mathcal{D}_\rho$. For any function $\hat{f} \in \mathcal{H}^2(\mathcal{D}_\rho)$ with $\rho > 1$, define $\mathcal{H}^2$-norm by*

$$\begin{aligned} \|\hat{f}\|_2 &= \left(\frac{1}{2\pi}\int_0^{2\pi} |\hat{f}(e^{j\omega})|^2\, dw\right)^{1/2} \\ \|\hat{f}\|_{2,\rho} &= \left(\frac{1}{2\pi}\int_0^{2\pi} |\hat{f}(\rho e^{j\omega})|^2\, d\omega\right)^{1/2} \end{aligned} \quad (4)$$

Then, $\hat{f} \in \mathcal{H}^\infty$, and

$$\|\hat{f}\|_\infty \leq \sqrt{\frac{\rho+1}{\rho-1}}\|\hat{f}\|_2^{1/2}\|\hat{f}\|_{2,\rho}^{1/2}.$$

Proof: It is not difficult to show that

$$\hat{f}(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(\sqrt{\rho}e^{jt})(1-\rho^{-1})}{1-\rho^{-1/2}\cos(\omega - t)+\rho^{-1}}\, dt$$

by Possion's integral. Hence, Schwartz inequality implies that

$$\begin{aligned} \|\hat{f}\|_\infty^2 &\leq \|\hat{f}\|_{2,\sqrt{\rho}}^2 \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{1-\rho^{-1}}{1-\rho^{-1/2}\cos(t)+\rho^{-1}}\right)^2 dt \\ &= \sqrt{\frac{\rho+1}{\rho-1}}\|\hat{f}\|_{2,\sqrt{\rho}}. \end{aligned}$$

From [9], any function $\hat{g} \in \mathcal{H}^2(\mathcal{D}_{r_2})$ satisfies inequality

$$\|\hat{g}\|_{2,r}^{\log(r_2/r_1)} \leq \|\hat{g}\|_{2,r_1}^{\log(r_2/r)}\|\hat{g}\|_{2,r_2}^{\log(r/r_1)}$$

provided that $r_1 < r < r_2$. Taking $r_1 = 1$, $r = \sqrt{\rho}$ and $r_2 = \rho$ for $\hat{g} = \hat{f}$, it follows that

$$\|\hat{f}\|_\infty \leq \sqrt{\frac{\rho+1}{\rho-1}}\|\hat{f}\|_2^{1/2}\|\hat{f}\|_{2,\rho}^{1/2}$$

The lemma is thus true. ∎

A final lemma which will be used is the n-width in approximation theory [15]. Define $\mathcal{P}_m$ as the collection of all $(m-1)$th order causal polynomial (or FIR) model

$$\mathcal{P}_m = \left\{\hat{p} : \hat{p} = p_0 + p_1 z + \ldots + p_{m-1}z^{m-1}\right\}. \quad (5)$$

The following lemma can be found in [15] (Theorem 2.1 of page 250).

Lemma 2.3 *Let $M > 0$, $\rho > 1$. Then for $m = 0, 1, \ldots,$*

$$\sup_{\hat{h}\in\mathcal{H}(M,\rho)}\inf_{\hat{p}\in\mathcal{P}_m} \|\hat{h} - \hat{p}\|_\infty = M\rho^{-m}.$$

Further, for any $\hat{h} \in \mathcal{H}(M, \rho)$, the globally optimal approximant of $\hat{h}$

$$\hat{p}_m^*[\hat{h}] := \sum_{k=0}^{m-1}\left(1-\rho^{2(k-m)}\right)h_k z^k,$$

achieves the bound $M\rho^{-m}$. That is,

$$\sup_{\hat{h}\in\mathcal{H}(M,\rho)} \|\hat{h} - \hat{p}_m^*[\hat{h}]\|_\infty = M\rho^{-m}.$$

3 Main Results

In this section, least square based linear algorithms in [9] will be revisited. The purpose of this paper is to give a unified treatment of the linear algorithms in [9] and to derive improved error bounds applicable to the nonuniformly spaced frequency response data samples.

Let the experimental data sequence $\{E_k^N\}$ be given in (2), with its DFT coefficient denoted by

$$c_k(E^N) = \frac{1}{N}\sum_{i=0}^{N-1} E_{i+1}^N W_N^{ik}, \quad W_N = e^{j2\pi/N}, \; k = 0, 1, \ldots, N-1.$$

Let the identified model be denoted as

$$\hat{h}_{id}(z) = \sum_{k=0}^{n-1} p_k z^k, \; n \leq N.$$

The objective of the linear algorithms is to determine p_k's which are linear functions of the experimental data (2) such that the identification error measured in $\mathcal{H}^\infty$ norm is suitably small. The class of linear algorithms in [9] are based on the solutions of certain least square problems. Two such least square problems which lead to linear algorithms are constrained minimizations

$$J_1 = \min_{p_k, \|\hat{h}_{id}\|_{2,\rho}\leq M}\left[\sum_{k=0}^{N-1}\left|p_k - c_k(E^N)\right|^2\right]^{1/2}$$

with the convention that $p_k = 0$ for $n \leq k \leq N-1$ if $n < N$, and

$$J_2 = \min_{\hat{h}_{id}^n\in\mathcal{P}_n, \|\hat{h}_{id}\|_{2,\rho}\leq M}\left[\frac{1}{N}\sum_{k=0}^{N-1}\left|\hat{h}_{id}(W_N^k) - E_{k+1}^N\right|^2\right]^{1/2}$$

These two minimization problems are treated differently in [9] and result in two different linear algorithms, as well as error bounds. It turns out that the two different linear algorithms produce same identified model.

Proposition 3.1 *The two different constrained minimization problems J_1, and J_2 yield same solution $\hat{h}_{id}$, and $J_1 = J_2$.*

Proof: The fact that $J_1 = J_2$ is a direct consequence of the mixed Parserval's theorem in Lemma 2.1. Since the constrained minimizations are least square problems, they have unique solution. The proposition is thus true. ∎

The above observation is important, since one needs to consider only J_2. The next result gives an improved bound for the resulting identification error.

Theorem 3.2 *Let the noisy experimental frequency response data be given in (2). Let the identified model be $\hat{h}_{id}(z) \in \mathcal{P}_n$ where the coefficients are defined by the solution of the constrained minimization problem J_2. Then, the worst-case identification error satisfies*

$$e_N(M,\rho,\epsilon) \leq M\rho^{-n} + 2M\sqrt{\frac{\rho+1}{\rho-1}\left(\frac{\epsilon}{M}+\rho^{-n}\right)}.$$

Proof: The minimization problem given has a solution with

$$\left[\frac{1}{N}\sum_{k=0}^{N-1}\left|\hat{h}_{id}(W_N^k) - E_k^N\right|^2\right]^{1/2} \leq \epsilon + M\rho^{-n}$$

To see this note that $\hat{h}_{id} = \hat{p}_n^*[\hat{h}]$ yields one such solution by the n-width approximation. Hence,

$$\left[\frac{1}{N}\sum_{k=0}^{N-1}\left|\hat{h}_{id}(W_N^k) - \hat{h}(W_N^k)\right|^2\right]^{1/2} \leq 2\epsilon + M\rho^{-n}$$

by the hypothesis on E^N, which in turn implies that

$$\begin{aligned}\left\|\hat{h}_{id} - \hat{p}_n^*[\hat{h}]\right\|_2^2 &= \frac{1}{N}\sum_{k=0}^{N-1}\left|\hat{h}_{id}(W_N^k) - \hat{p}_n^*[\hat{h}](W_N^k)\right|^2 \\ &\leq 4(\epsilon + M\rho^{-n})^2\end{aligned}$$

in light of Lemma 2.1. Note also that

$$\|\hat{p}_n^*[\hat{h}]\|_{2,\rho} \leq \|\hat{h}\|_{2,\rho} \leq M, \forall \hat{h} \in \mathcal{H}(M,\rho).$$

Hence, $\|\hat{h}_{id} - \hat{p}_n^*\|_{2,\rho} \leq 2M$. Using Lemma 2.2, it follows that

$$\|\hat{h}_{id} - \hat{p}_n^*[\hat{h}]\|_\infty \leq 2M\sqrt{\frac{\rho+1}{\rho-1}}\left(\frac{\epsilon}{M}+\rho^{-n}\right)^{1/2}$$

with routine algebra. Now the error bound can be established by noting that

$$e_N(M,\rho,\epsilon) \leq M\rho^{-n} + \sup_{\hat{h}\in\mathcal{H}(M,\rho),|\eta_k|\leq\epsilon}\left\{\|\hat{h}_{id}^n - \hat{p}_n^*[\hat{h}]\|_\infty\right\}.$$

∎

It should be clear that the error bound in Theorem 3.2 improves the one in [9], and has a simpler form. This is due to the fact that for the case $\epsilon = 0$, the error bound in Theorem 3.2 decays in the order $\mathcal{O}(\rho^{-n/2})$, whereas in [9], the error decays in the order $\mathcal{O}(\rho^{-\alpha n})$ where $\alpha < 1/2$ for $\rho > 1$. Further, the factor $(\rho+1)/(\rho-1)$ in [9] is replaced by $\sqrt{(\rho+1)/(\rho-1)}$ in Theorem 3.2.

Following the same steps as in [9], Theorem 3.2 gives the following tuned linear algorithm with an explicit identification error bound,

Corollary 3.3 *Let the noisy experimental frequency response data be given by (.?). Denote $\tilde{\epsilon} = \epsilon + M\rho^{-n}$. Form the identified model*

$$\hat{h}_{id}^n(z) = \sum_{k=0}^{n-1}\frac{c_k(E^N)}{1+\left(\frac{\tilde{\epsilon}}{M}\right)^2\rho^{2k}}z^k,\ n \leq N,$$

with $\{c_k(E^N)\}$ the DFT coefficient of $\{E_k^N\}$. Then, the worst case identification error satisfies

$$e_N(M,\rho,\epsilon) \leq M\rho^{-n} + (1+\sqrt{2})M\sqrt{\frac{\rho+1}{\rho-1}}\left(\frac{\epsilon}{M}+\rho^{-n}\right)^{1/2}.$$

Proof: Consider the following unconstrained minimization problem:

$$J = \min_{\hat{h}_{id}^n\in\mathcal{P}_n}\left(\frac{1}{N}\sum_{k=0}^{N-1}\left|\hat{h}_{id}^n(W_N^k) - E_k^N\right|^2\right) + \left(\frac{\tilde{\epsilon}}{M}\right)^2\|\hat{h}_{id}^n\|_{2,\rho}^2$$

with $\hat{h}_{id}^n = p_0 + p_1 z + \ldots + p_{n-1}z^{n-1}$. It admits a solution $\hat{h}_{id}^n$ such that $J \leq 2\tilde{\epsilon}^2$. This can be shown by taking $\hat{h}_{id}^n = \hat{p}_n^*[\hat{h}]$. With the solution for the above unconstrained minimization problem,

$$J_c = \left[\frac{1}{N}\sum_{k=0}^{N-1}\left|\hat{h}_{id}^n(W_N^k) - E_k^N\right|^2\right]^{1/2} \leq \sqrt{2}\tilde{\epsilon}.$$

It follows that

$$\begin{aligned}\|\hat{h}_{id}^n - \hat{p}_n^*[\hat{h}]\|_2 &\leq J_c + \left(\frac{1}{N}\sum_{k=0}^{N-1}\left|\hat{p}_n^*[\hat{h}](W_N^k) - E_k^N\right|^2\right)^{1/2} \\ &\leq (1+\sqrt{2})(\epsilon + M\rho^{-n}),\end{aligned}$$

and $\|\hat{h}_{id}^n - \hat{p}_n^*\|_{\infty,\rho} \leq M + \sqrt{2}M$. The worst case identification error bound can then be obtained following the steps in the proof of Theorem 3.2. To obtain the explicit solution $\hat{h}_{id}^n$, one notices that the unconstrained minimization problem is equivalent to

$$\begin{aligned}J = &\min_{p_k}\sum_{k=0}^{n-1}\left|p_k - c_k(E^N)\right|^2 \\ &+ \sum_{k=n}^{N-1}\left|c_k(E^N)\right|^2 + \left(\frac{\tilde{\epsilon}}{M}\right)^2\left(\sum_{k=0}^{n-1}|p_k|^2\rho^{2k}\right)\end{aligned}$$

by mixed Parseval's theorem. The optimal solution p_k's can then be obtained by setting the partial derivatives $\partial J/\partial p_k = 0$ which gives the solution

$$p_k = \frac{c_k(E^N)}{1+\left(\frac{\tilde{\epsilon}}{M}\right)^2\rho^{2k}},\ k = 0,1,\ldots, ?-1.$$

The proof is now completed. ∎

Before concluding this section, it should be emphasized that all the linear algorithms discussed so far are derived for uniformly spaced frequency response data. Naturally, one would like to know whether or not the linear algorithms can be adapted to the case where the frequency response data is nonuniformly spaced as studied in [1, 13]. This question will be answered for the tuned linear algorithm.

Corollary 3.4 *Let the experimental frequency response data E^N be obtained at $\{\omega_i\}_{i=0}^{N-1}$ which is not uniformly spaced. Define matrix U_l as*

$$U_l = \begin{bmatrix} 1 & e^{j\omega_0} & \ldots & e^{j(l-1)\omega_0} \\ 1 & e^{j\omega_1} & \ldots & e^{j(l-1)\omega_1} \\ \ldots & \ldots & \ldots & \ldots \\ 1 & e^{j\omega_{N-1}} & \ldots & e^{j(l-1)\omega_{N-1}} \end{bmatrix},$$

where $0 < l \leq N$. Suppose the identified model $\hat{h}_{id}^n \in \mathcal{P}_n$ is obtained from the following unconstrained minimization problem

$$J_n = \min_{\hat{h}_{id}^n\in\mathcal{P}_n}\left(\frac{1}{N}\sum_{k=0}^{N-1}\left|\hat{h}_{id}^n(e^{j\omega_k}) - E_k^N\right|^2\right) + \left(\frac{\tilde{\epsilon}}{M}\right)^2\|\hat{h}_{id}^n\|_{2,\rho}^2$$

with same $\tilde{\epsilon}$ as in Corollary 3.3. Then, the worst case identification error satisfies

$$e_N(M,\rho,\epsilon) \leq M\rho^{-n} + \frac{\sqrt{N}(1+\sqrt{2})M}{\sigma_{min}(U_N)}\sqrt{\frac{\rho+1}{\rho-1}\left(\frac{\epsilon}{M}+\rho^{-n}\right)}$$

Proof: It is noted that the tuned algorithm is similar to that in Corollary 3.3 except that the frequency response data is not uniformly spaced. Same argument in Corollary 3.3 gives $J_n \leq 2\hat{\epsilon}^2$, and thus a solution $\hat{h}_{id}^n \in \mathcal{P}_n$ exists such that

$$\left[\frac{1}{N}\sum_{k=0}^{N-1}\left|\hat{h}_{id}^n(e^{j\omega_k}) - \hat{p}_n^*(e^{j\omega_k})\right|^2\right]^{1/2} \leq (1+\sqrt{2})\hat{\epsilon}.$$

Denote $\hat{f} = \hat{h}_{id}^n - \hat{p}_n^*[\hat{h}] \in \mathcal{P}_n$. Then, the above is equivalent to

$$\left[\frac{1}{N}\sum_{k=0}^{N-1}\left|\hat{f}(e^{j\omega_k})\right|^2\right]^{1/2} = \frac{1}{\sqrt{N}}\|U_N F\|_2 \leq (1+\sqrt{2})\hat{\epsilon}$$

where F is a column vector of size N with first n elements being the n coefficients of $\hat{f}$ respectively, and the rest. elements zero. Note that by the definition of singular values,

$$1 \leq \sqrt{N}\sigma_{max}(U_N^{-1}) = \frac{\sqrt{N}}{\sigma_{min}(U_N)}.$$

It follows that

$$\begin{aligned}\|\hat{h}_{id}^n - \hat{p}_n^*[\hat{h}]\|_2 &= \|F\|_2 \leq \sqrt{N}(1+\sqrt{2})\hat{\epsilon}\sigma_{max}(U_N^{-1}) \\ &= \frac{\sqrt{N}(1+\sqrt{2})\hat{\epsilon}}{\sigma_{min}(U_N)}\end{aligned}$$

for any $\hat{h} \in \mathcal{H}(M, \rho)$. Similarly, $\|\hat{h}_{id}^n\|_{2,\rho} \leq \sqrt{2}M$. Thus

$$\|\hat{h}_{id}^n - \hat{h}\|_{2,\rho} \leq (1+\sqrt{2})M \leq \frac{\sqrt{2}(1+\sqrt{2})M}{\sigma_{min}(U_N)}.$$

Hence, the error bound can be established following the same steps in the proof of Corollary 3.3.

∎

A few comments are in order. First, denote P as a column vector of size n with coefficients of h_{id}^n as the elements. Then, the minimization problem in Corollary 3.4 has a matrix representation

$$\min_{P \in R^n}\left\|\begin{bmatrix}\frac{1}{\sqrt{N}}U_n \\ \frac{\epsilon}{M}V_m\Lambda\end{bmatrix}P - \begin{bmatrix}\frac{1}{\sqrt{N}}\mathcal{E} \\ 0\end{bmatrix}\right\|_2^2$$

where U_n is same as in Corollary 3.4 with $l = n$, $\Lambda = \text{diag}(1, \rho, ..., \rho^{(n-1)})$, and

$$V_m = \frac{1}{\sqrt{m}}\begin{bmatrix}1 & 1 & \cdots & 1 \\ 1 & W_m^1 & \cdots & W_m^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & W_m^{m-1} & \cdots & W_m^{(m-1)(n-1)}\end{bmatrix}$$

$$\mathcal{E} = \begin{bmatrix}E_0^N \\ E_1^N \\ \cdots \\ E_{N-1}^N\end{bmatrix}, \quad m \geq n. \quad (6)$$

Hence, the solution P is easily obtained from the orthogonality condition. Second, since V_m satisfies $V_m^H V_m = I_n$ for $m \geq n$ where V_m^H denotes the conjugation transpose of V_m,

$$P = \left(\frac{U_n^H U_n}{N} + \frac{\hat{\epsilon}^2}{M^2}\Lambda^2\right)^{-1}\frac{U_n^H \mathcal{E}}{N} \quad (7)$$

which is well defined even if $U_n^H U_n$ in singular, though in this case, the error bound is not defined. It should be clear that $U_n^H U_n$ is nonsingular if and only if all $\{\omega_i\}$'s are distinct. Finally, note that if ω_i's are uniformly spaced, then $U_n^H U_n = N I_n$ for which $U_n^H \mathcal{E}/N$ is the inverse DFT of the experimental measurement data and both the solution and the error bound reduce to those of Corollary 3.3 by noting that $\sigma_{max}(U_N) = \sigma_{min}(U_N) = \sqrt{N}$.

4 Applications to Identification of Flexible Structures

It is known that two-stage nonlinear algorithms [s, 4, 14] are not effective for lightly damped systems such as flexible structures [5]. The difficulty lies in the model reduction part of the identified model. Most model reduction algorithms such as balanced realization and Hankel norm approximation require computations of controllability and observability gramians. Since the identified model from the two-stage nonlinear algorithm inevitably has a high order for flexible structures and is a sum of a rational function (resulted from Nehari approximant) and a causal polynomial function, it is almost impossible to compute controllability and observability gramians, or the resulting gramians are not accurate. This problem also exists for interpolation based algorithms [2, 7]. It is noted that the linear algorithm studied in this paper produce identified models having FIR structure, the computation for controllability and observability gramians requires only one singular value decomposition [6]. Due to the reliability of singular value decomposition, the reduced order model retains dominant modes of the flexible structure. Hence, the least square linear algorithm provides an alternative which is quite effective as shown in next section. Further, the modeling error caused by model reduction can be easily bounded using the existing results.

5 Illustrative Examples

To illustrate the effectiveness of the linear algorithm, two examples are used. The true system of the first example is given by

$$\hat{h}(z) = \frac{10}{z^2 + 5z + 10} \quad (8)$$

It can be verified that $\hat{h}(z)$ with $M = 2.1$ and $\rho = 1.5$. The experimental data is generated by uniform samples of $\hat{h}(e^{j\omega})$ with corrupting noise $\hat{\eta}_k = \epsilon e^{j\theta}$ where θ is the uniformly distributed random variable, We have chosen $\epsilon = 0.2$ which is roughly one-tenth of the $\|\hat{h}\|_\infty$. The simulation consists of $N = 64$ experimental data points for both uniform and nonuniform sampling cases. The identified models are obtained using least square based linear algorithm. The magnitude error responses are plotted in Figure 1 with solid line for uniformly spaced sampling case and with dashed line for nonuniformly spaced sampling case. Because nonuniform sampling takes more samples at fast variation interval and fewer samples at slow variation interval of the frequency response, it often has a better performance than that of uniform sampling although it has a larger error bound.

The second example is taken from [5] where the true system is a flexible structure. While the two-stage nonlinear algorithm in [8, 4, 14] is not effective for identification of lightly damped systems [5], the linear algorithm studied in this paper, in conjunction with the balanced model reduction, yields a very impressive result. The magnitude response of the experimental data is plotted in Figure 2 with solid line. Since no *a priori* information on M, ρ and ϵ are given in [5], we have used $M = 130$, $\rho = 1.01$ and $\epsilon = 0.5$. Corollary 3.3 is applied with $N = n = 1024$ to obtain the FIR model. Balanced realization method for model reduction is then applied to obtain a low order model of McMillan degree 26. Its magnitude error response is plotted in Figure 3. We would like to comment that although other algorithms can also obtain similar results with less computational effort, the linear algorithm studied in this paper combined with the balanced model reduction constitutes a convergent algorithm for identification in H_∞. Since balanced model reduction for an FIR model needs to compute only one singular value decomposition of a 1024 x 1024 positive definite matrix [6], this example demonstrates that the algorithm is quite reliable.

G Conclusion

The least square based linear algorithms in [9] are revisited and new error bounds are derived. It is shown that the *tuned* linear algorithms in [9] are applicable to nonuniformly spaced frequency response data which are quite different from the two-stage nonlinear algorithms as in [1]. In particular, exponential convergence for noise free case is preserved.

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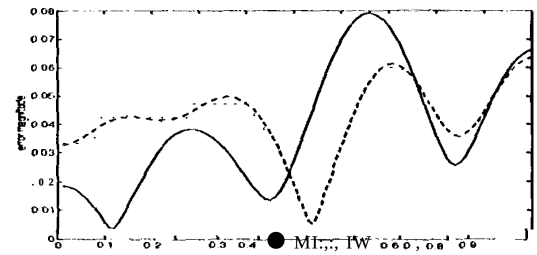

Figure 1: Magnitude response of the identification error for the first example

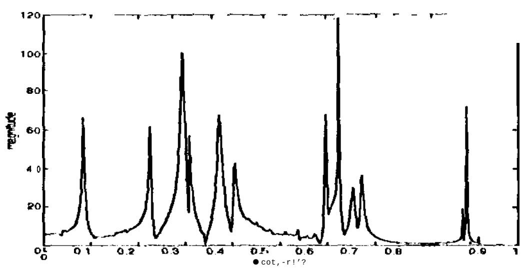

Figure 2: Magnitude response of the true system for the second example

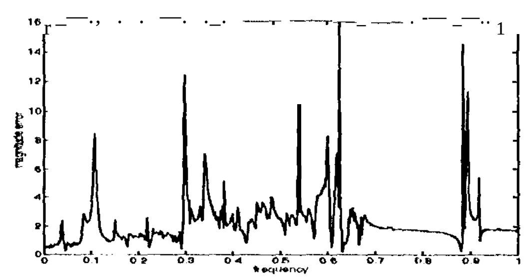

Figure 3: Magnitude response of the identification error for the second example