A GLOBALLY OPTIMAL MINIMAX SOLUTION FOR SPECTRAL OVERBOUNDING AND FACTORIZATION

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ABSTRACT

In this paper, an algorithm is introduced to find a minimum phase transfer function of specified order whose magnitude 'tightly' overbounds a specified real-valued nonparametric function of frequency. This method has direct application to transforming nonparametric uncertainty bounds (available from system identification experiments and/or plant modeling) into parametric representations required for modern robust control design software (i.e., a minimum-phase transfer function multiplied by a norm-bounded perturbation).

1. INTRODUCTION

Assume that a discrete-time plant $P(z^{-1})$ is estimated as $\hat{P}(z^{-1})$, and let L denote the uncertainty in the estimate. For example, three **common** characterizations of plant uncertainty are LA-additive uncertainty, L_I-input multiplicative uncertainty, and Lo-output multiplicative uncertainty, where (Morari and Zafiriou [17] page 224),

$$
\begin{aligned}
L_A &= P - \hat{P} \\
L_I &= \hat{P}^{-1}(P - \hat{P}) \\
L_O &= (P - \hat{P})\hat{P}^{-1}
\end{aligned}
\tag{1}
$$

Note that multiplicative representations require a square plant. Let L denote any one of the above three quantities, Suppose, a nonparametric overbound $\ell(\omega)$ on L is known such that,

$$\ell(\omega) > \overline{\sigma}(L(e^{-j\omega T}))\ for\ all\ \omega \in [0,\ \pi/T] \tag{2}$$

where T is the sampling period and $\overline{\sigma}(L)$ is the maximum singular value of L. Various methods are available to find $\ell(\omega)$ from raw data (cf., [3][11] [13] [16]). However, $\ell(\omega)$ is a nonparametric function of frequency and cannot be used directly in modern robust control software packages such as the Matlab Robust Control Toolbox [7], and μ synthesis software [2]. Instead, the uncertainty must be represented as a minimum phase transfer function matrix $\mathcal{W}(z^{-1})$ of a specified order such that,

$$L(e^{-j\omega T}) = \Delta \mathcal{W}(e^{-j\omega T}) \tag{3}$$

where A is norm-bounded, i.e.,

$$||\Delta||_\infty < 1$$

The choice of $\mathcal{W}$ in (3) can be structured or unstructured. For present purposes,' the simplest choice is to usc a scalar matrix representation

$$\mathcal{W} = W \cdot I \tag{4}$$

where W is a single-input single-output rational function,

In order to incorporate the uncertainty bound (3) into a robust control design, a systematic method for determining the weighting W in (4) is needed. Ideally, any approach to find W should satisfy the following properties:

(PI) W must overbound the uncertainty ℓ, i.e.,

$$|W(e^{-j\omega T})| \geq \ell(\omega) \ for\ all\ \omega \in [0, \ \pi/T]$$

in order to ensure the existence of some $||\Delta||_\infty$ <1 satisfying (3).

(P2) W should be as *tight* an overbound as possible to avoid conservatism in the final robust cent rol design.

(P3) W should bc of specified order (in fact as low order as possible) since it will be incorporated as a weighting and increase the final controller order.

(P4) W should be stable and minimum-phase.

These requirements on W rule out using several powerful methods from the complex analysis literature. For example, interpolation methods such as Nevanlinna-Pick theory for finding rational interpolants to complex valued data [20] c1o not address the real-valued data case, and do not satisfy the above properties. For the same reasons, the extension of interpolatory methods to the noisy data case (cf., Helmicki et al. [13] [14], Gu and Kargonekar [12], Chen et al. [6], Partington [18], etc.) are not directly applicable to the present problem. Even if interpolatory methods could be appropriately modified to overbound real-valued data sets (i.e., PI) and do so in a minimax optimal sense (i.e., P2), the final rational fits tend to be of very high order (e.g., on the order of the number of data points) and would not satisfy properties P3 and P4 in general.

In this paper, a mathematical programming approach is used to find a W which satisfies all of the properties P1-P4. The main idea is to reformulate the problem so as to find a *spectrally factorizable* rational function W^* W whose magnitude tightly overbounds the *squared* data 4^2(w). This is done in Sect, 2 by posing a minimax nonlinear optimization problem to ensure tightness-of-fit of a rational function having specified order, with side constraints to ensure that the data is overbounded, and that the solution admits a spectral factorization. A key result in Sect. 3, is that the nonlinear optimization problem can be solved by a sequence of reweighted constrained linear problems. In particular, a globally convergent Linear Programming Spectral Overbounding and Factorization (LPSOF) algorithm is presented based on solving a sequence of linear programming problems [19]. The LPSOF algorithm provides a globally optimal solution to the nonlinear problem,

2. PROBLEM FORMULATION

In this section, a nonlinear constrained optimization is posed to compute a minimum-phase transfer function W of order m such that $|W|$ is a tight overbound on $\ell(\omega)$ for all ω. With this result, the uncertainty can bc written in standard form $L = \Delta W$ where $||\Delta||_\infty <1$. Forming the quantity $W(z)W(z^{-1})$ and evaluating on the unit circle gives an expression of the form,

$$W^* W = \frac{\&1-}{\alpha(\omega)} \tag{5}$$

where,

$$\beta(\omega) =@, + \beta_1 cos(\omega T) + .,. +- \beta_m cos(m\omega T) \tag{6a}$$

$$\alpha(\omega) = 1 + \alpha_1 cos(\omega T) +...+ \alpha_m cos(m\omega T) \tag{6b}$$

It is noted that $\alpha(\omega)$ is defined as monic without loss of generality.

Constraints for **Overbounding**

The requirement that $|W|$ bc an overbound on $\ell(\omega)$ is equivalent to the requirement that $|W|^2$ is an overbound on ℓ^2 and can bc expressed as,

$$\frac{\beta(\omega)}{\alpha(\omega)} \geq \ell^2(\omega) \quad for \quad all \quad \omega \in [0, \pi/T] \tag{7}$$

Constraints for **Tight** Overbounding

The requirement that $|W|^2$ be a "tight" overbound can be expressed as,

$$\min_{\delta,\alpha,\beta} \delta \tag{8}$$

where,

$$\delta \geq= \{ (\frac{\beta(\omega)}{\alpha(\omega)} - \ell^2(\omega))q^{-1}(\omega)\} \quad for \quad all \quad \omega \in [0, \pi/T] \tag{9}$$

Here, the criterion minimizes a worst-case error δ, which is frequency weighted by the quantity $q^{-1}(\omega)$.

Constraints for Spectra] **Factorizability**

The requirement that the overbound β/α admits a spectral factorization can be satisfied by ensuring that (Astrom [1])

$$\beta(\omega)/\alpha(\omega) > 0 \; for \; all \; \omega \in [0, \pi/T] \tag{10a}$$

$$\alpha(\omega) > 0 \; for \; all \; w \in [0, \pi/T] \tag{10b}$$

Note that condition (10a) is implied by (7), and condition (10b) can be enforced explicitly by the constraint,

$$\alpha(\omega) \geq \underline{\alpha} > 0 \;\; for \;\; all \;\; w \in [0, \pi/T] \tag{11*}$$

for some small $\underline{\alpha}$. For technical reasons, it will be convenient to enforce a similar constraint on β as

$$\beta(\omega) \geq \underline{\beta} > 0 \;\; for \;\; all \;\; \omega \in [0, \; \pi/T] \tag{11b}$$

for some small $\underline{\beta}$.

In summary, it is desired to solve the optimization problem (8)(9) for α, β subject to constraints (7) and (1 0a,b).

3. THE LPSOF ALGORITHM

In this section, the LP-Spectral Overbounding and Factorization (LPSOF) algorithm is introduced which solves the constrained nonlinear optimization problem of Sect. 2 on grid of points A $= \{\omega_1, \ldots . \omega_n\}$. Modifications to extend these results to *all* $\omega \in [0, \pi/T]$ will also be discussed.

The constrained optimization problem restricted to points of the set A can be written as,

$$\min_{\delta, \alpha_j, \beta_j} \delta \tag{12}$$

subject to

$$\beta(\omega_i) - \ell^2(\omega_i)\alpha(\omega_i) \geq 0 \tag{13a}$$

$$\beta(\omega_i) - \ell^2(\omega_i)\alpha(\omega_i) \leq \delta q(\omega_i)\alpha(\omega_i) \tag{13b}$$

$$\beta(\omega_i) \geq \underline{\beta}; \; \alpha(\omega_i) \geq \underline{\alpha} \tag{13c}$$

$$for \; all \; \omega_i, \; i = 1, \ldots, n$$

where $\alpha(\omega)$ and $\beta(\omega)$ are defined by (6a) b). A key observation from (12)(13) is that for fixed δ the optimization over α, β is simply a linear programming problem to find a *feasible solution* for the coefficients $\alpha_i \beta_i$. Hence, the joint optimization problem can be solved by a nested search procedure where an outer-loop systematically decreases δ while an inner-loop finds feasible solutions in the variables α and β for fixed δ. The procedure terminates when the smallest γ is found which admits a feasible solution. This approach is denoted as the **LP-Spectral Overbounding** and Factorization **(LPSOF)** Algorithm.

To solve the problem (12-13), one must begin with upper and lower bounds for the optimal value δ. For example, one can choose the lower bound $\delta_- =$ O and let the upper bound δ_+ be derived from some starting feasible suboptimal solution (an obvious choice is $\alpha = 1$, $\beta = \max_{\omega_i} \ell^2(\omega_i)$). Then $\hat{\delta} = (\delta_+ + \delta_-)/2$, becomes an updated value for δ_+ or δ_- depending on whether or not the inequalities (13) can be satisfied for $\delta = \hat{\delta}$ (i.e., the bisection method [8]). In this way the LPSOF algorithm converges to the optimal value of δ geometrically, (i.e., as a power of 1/2).

This process can be further accelerated by effectively linearizing about the candidate value $\hat{\delta}$. Thus, given $\hat{\delta}$, one can solve the linearized problem:

$$\max_{u,\alpha_j,\beta_j} u \tag{14}$$

subject to

$$\beta(\omega_i) - \ell^2(\omega_i)\alpha(\omega_i) \geq 0 \tag{15a}$$

$$\beta(\omega_i) - \ell^2(\omega_i)\alpha(\omega_i) - \hat{\delta}q(\omega_i)\alpha(\omega_i) \leq -2 \ 1 \tag{15b}$$

$$\beta(\omega_i) \geq \underline{\beta} \qquad \alpha(\omega_i) \geq \underline{\alpha} \tag{15c}$$

$$for \ all \ \omega_i, i = 1, \ldots, n$$

Then $\hat{\delta}$ provides an update for δ_- or δ_+ according to whether the solution u is negative or nonnegative. In the latter case a sharper *a posteriori* estimate for δ_+ is derived via

$$\delta_+ = \max_{\omega\epsilon\Lambda} \left\{ \left(\frac{\beta(\omega)}{\alpha(\omega)} - \ell^2(\omega) \right) q^{-1}(\omega) \right\} \tag{16}$$

where $\alpha(\omega)$ and $\beta(\omega)$ are the solutions derived from (14)(1 5). It is also worth noting that setting $\hat{\delta}$ = O and solving (14)-(16) provides an excellent starting value for δ_+ to initialize the algorithm.

Remark 1 It is noted that the weighting g may be chosen as certain functions of the unknown polynomials α and β without violating the linear form of the constraints. Generally q(w) may be taken in the form

$$q(\omega) = q_0(\omega) + q_1(\omega)\frac{\beta(\omega)}{\alpha(\omega)} \tag{17}$$

where go(w) and q_1 (w) are positive for $\omega\epsilon\Lambda$ and specified beforehand, ■

Remark 2 By the fundamental properties of linear programming [9] [10], the LPSOF Algorithm is globally convergent and achieves a globally optimal solution to the discrete problem (12)(13). ■

Remark 3 For each fixed value of δ the semi-infinite linear programming problem (7)(9) and (11) (i. e., for continuous valued w $\in [0, \pi/T]$), can be solved as a sequence of discretized linear programming problems of the form (13) in the limit as the mesh becomes sufficiently fine (cf., [15], Sect. 7,2). Thus, one can recover the solution to the semi-infinite spectral overbounding and factorization problem (7)(8)(9)(11) using the LPSOF approach, by solving a sequence of linear programs with decreasing mesh spacing, for each fixed value of δ. However, decreasing the mesh spacing may not be desirable in practice, and alternative methods for approximating the solution to the semi-infinite problem will be presented in Sect. 4. ●

4. MODIFICATIONS OF THE LPSOF ALGORITHM

Strictly speaking, the LPSOF algorithm only enforces inequalities (7)(10a,b) at the points in the grid A. Hence, the inequalities may be violated *in-between* grid points, and the solution may not be a true overbound and/or may not admit a spectral factor, If this happens in practice, the simplest solution is generally to choose a denser grid (see Remark 3), and/or increase lower bounds $\underline{\alpha}, \underline{\beta}$ in (13c). However, there may be certain cases where these approaches are not desirable. Hence, in this section, systematic modifications of the LPSOF algorithm are presented to overcome this problem.

l?or convenience to subsequent discussion, we make the following assumptions,

Assumption 1: Let $\ell^2(\omega)$ be a linear splint interpolant to the points $\ell^2(\omega_i)$ defined on the grid A = $\{\omega_1, ..., \omega_n\}$ with piecewise linear segments having maximum slope κ, and maximum grid size $h = \max_i \{\omega_{i+1} - \omega_i\}$.

Assumption 2: The set of vectors $\{[\cos(k\omega_1 T), \ldots \cos(k\omega_n T)] \in \Re^n, k = 0,..,m\}$ arc linearly independent,

Modifications of the LPSOF algorithm to ensure proper behavior between grid points, fall into two categories, a-priori and a-posteriori. These methods will be discussed separately below.

A-Priori Modifications

The basic idea behind the a-priori modifications is to enforce additional linear constraints in the LPSOF algorithm so that the derivatives $\alpha'(\omega) = \frac{d}{d\omega}$ (o(w)), $\beta'(\omega) = \frac{d}{d\omega}(\beta(\omega))$ and $(\beta/\alpha)' = \frac{d}{d\omega}(\beta/\alpha)$ are suitably bounded *for all* $\omega \in [0, \pi/T]$. This clearly restricts the excursions of α, β and β/α *in-between grid points, so* that under Assumption 1, and specification of lower bounds $\underline{\alpha}, \underline{\beta}$ in (13c), the desired inequalities (7),(10a,b) can be satisfied.

Some useful definitions arc in order: A function $x(\omega)$ defined on the interval w $\in \Omega$ is said to be *uniformly bounded from above* if $|x(\omega)| \leq$ C < ∞ for all w $\in \Omega$. Here, the quantity C is denoted as the *uniform upper bound.* Similarly, the function $x(\omega)$ is said to be *uniformly bounded from below* if $|x(\omega)| \geq$ c > 0 for all $\omega \in \Omega$. The quantity c is denoted as the uniform *lower bound.*

A method to uniformly bound the aforementioned derivatives is now introduced. For some positive $\overline{\alpha}$, K, K_α, and K_β, let the following linear constraints be imposed on the grid A:

$$\alpha(\omega_i) \leq \overline{\alpha} \tag{18a}$$

$$\beta(\omega_i) \leq K\alpha(\omega_i) \tag{18b}$$

$$|\alpha'(\omega_i)| \leq K_\alpha \, |\alpha(\omega_i)| \tag{18c}$$

$$|\beta'(\omega_i)| \leq K_\beta \, |\beta(\omega_i)| \tag{18d}$$

If linear constraints (1 8a-d) are used to augment linear constraints (13a-c) of the LPSOF algorithm, it can be shown that,

$$\underline{\alpha} \leq \alpha(\omega_i) \leq \overline{\alpha} \tag{19a}$$

$$\underline{\beta} \le \beta(\omega_i) \le K\overline{\alpha} \tag{19b}$$

$$|\alpha'(\omega_i)| \le K_\alpha \overline{\alpha} \tag{19c}$$

$$|\beta'(\omega_i)| \le K_\beta K \overline{\alpha} \tag{19d}$$

$$\left| \frac{d}{d\omega} \frac{\beta(\omega)}{\alpha(\omega)}|_{\omega=\omega_i} \right| \le K(K_\beta + K_\alpha) \tag{19c}$$

where (19C) follows from,

$$\left(\frac{\beta}{\alpha}\right)' = \frac{\beta'}{a'} - \frac{\alpha'}{\alpha^2} = \frac{\beta}{\alpha}\left(\frac{\beta'}{\beta} - \frac{\alpha'}{\alpha}\right) \tag{20}$$

Inequalities (19a-c) imply that $\alpha, \beta, \beta/\alpha$ and their derivatives are bounded on the grid A. Under Assumption 2, these bounds at the grid points impose bounds on the coef-ficients α_i, β_i, (this is because the matrix of trigonometric functions which determines these coefficients has a bounded inverse, cf., Dahlquist and Bjork [8])). This implies that $|\alpha(\omega)|, |\beta(\omega)|, |\alpha'(\omega)|$ and $|\beta'(\omega)|$ arc uniformly bounded from above since they are bounded functions of the bounded coefficients α_i, β_i. If the uniform upper bounds **on** $|\alpha'(\omega)|$ and $|\beta'(\omega)|$ are sufficiently small, the constraints (13c) on the grid points imply the existence of uniform (non-zero) lower bounds on $|\alpha(\omega)|, |\beta(\omega)|$. Hence, by systematically decreasing the values of K_β and K_α (with the other constraints fixed) onc can make $|\alpha'(\omega)|$ and $|\beta'(\omega)|$ arbitrarily small, and there will always be some point at which the inequalities (1 0a,b) are satisfied uniformly in w. Given that (10a,b) is satisfied in this manner, it follows from (20) that $|(\beta/\alpha)'|$ is bounded uniformly from above. Using (13a) and Assumption 1, it follows that (7) is satisfied when this uniform upper bound on 1(/?/0)'] falls below the value of κ defined in Assumption 1,

In summary, by augmenting the linear constraints (13a-c) of the LPSOF algorithm by linear constraints (18a-d), and solving a sequence of problems where K_β and K_α arc systematically decreased, there will bc a point at which (7)(10a,b) are satisfied ensuring that β/α is an overbound on ℓ^2 *for all* $w \in [0, \pi/T]$, and spectrally factorizable.

A-Posteriori Modifications

The basic idea behind a-posteriori mollifications, is to slightly perturb the unmodified LPSOF solution so that inequalities (7),(10a,b) arc satisfied uniformly in ω. For grid A sufficiently fine, a small perturbation can always be found which dots the job. To sec this, note that the solution to the problem (12-13) will always satisfy bounds of the form (19), if the quantities $\overline{\alpha}$, K, K_α, and K_β are computed *a-posteriori*. As noted in the previous discussion, Assumption 2 guarantees corresponding bounds for the coefficients α_i, β_i which in turn impose uniform upper bounds on $|\alpha(\omega)|, |\beta(\omega)|, |\alpha'(\omega)|$ and $|\beta'(\omega)|$. Then for a sufficiently small grid size h, uniform constraints (7)(10a,b) can be satisfied by means of an O(h) perturbation of $\alpha(\omega)$, f?(w), and δ.

The construction above implicitly assumes that $\bar{\alpha}$, K, K_α, and K_β are reasonably sized. If not, these quantities can be explicitly constrained *a-priori*, as done earlier.

5. NUMERICAL EXAMPLE

In this section, the LPSOF algorithm is used to determine spectrally factorizable overbounds on additive uncertainty estimates obtained from large space structure identification experiments [5]. Raw additive uncertainty data $\ell^2(\omega_i)$ adapted from [5] is shown in Fig. 1, depicted by the symbol '*', on the grid A $=\{\omega_i$ = in/(128 $T), i=$ 1, ..., 128}, where $T=$ .05 seconds. In addition to the raw data set, the envelope $\bar{\ell}^2$ is depicted in Fig. 1. The envelope $\bar{\ell}^2$ is a smoothed nonparametric overbound on the raw data ℓ^2.

In all subsequent examples, the LPSOF algorithm of Sect. 3 is used in sequential linearized form (14)(15), with underbounds $\underline{\alpha}=\underline{\beta}=$ O and frequency weighting $q(\omega)= q_0+q_1\beta(\omega)/\alpha(\omega)$ where $q_1=1$, and q_0 remains to be specified (see (17)). The dual rather than primal form of the underlying LP problem [9] [10] is implemented to considerably reduce the number of constraint equations. In all runs, 10-15 iterations were sufficient to ensure convergence of δ to 6 significant digits.

Overbounding Raw Data

In this section, the raw data ℓ^2 of Fig.1 is overbounded using the LPSOF algorithm of Sect. 3..

The first set of runs is generated by fixing $m=4$ and $q_1=1$ and varying weighting factor q_0 as $q_0=$ *O*, .01, .1, 1. The results arc summarized in Fig. 2, where the raw data ℓ^2 is depicted by the symbol '*). It is seen that the overbound associated with $q_0=$ O dots a reasonably good job of overbounding the data, but tends to sacrifice some accuracy in the peaks for accuracy in the troughs (e.g., there is about an 8 db overshoot of the main peak). Methods to improve the results become apparent,

Met hod 1: De-emphasize the weighting of trough data by increasing go.

Method 2: Reduce oscillatory behavior of the data by overbounding the envelope $\bar{\ell}^2$ rather than raw data ℓ^2.

Method 1 motivates the remaining runs of Fig. 2 which are generated by increasing q., (Method 2 motivates using envelope data rather than raw data, which will be discussed later). It is seen from Fig, 2. that increasing q_0 from O to .01 improves the accuracy of the overbound by several db in the vicinity of the peaks at the cost of accuracy in the troughs. Increasing go further to .1 and 1 continues this trend. Hence, the user can choose from the family of curves in Fig, 2, to trade-off accuracy in the peaks for accuracy in the troughs.

The second set of runs is generated by fixing q_1 = 1, *go = 1* and varying the order m *as* $m=$ 2, 4, 6. The results are summarized in Fig. 3. It is seen that the successive overbounds improve uniformly as the order is increased. In particular, the peaks are fitted reasonably well by bounds of all orders (this is a consequence of using q_0 = 1 for all runs), while most of the improvement from increasing order comes from fitting the troughs.

Overbounding Envelope Data

In this study (in accordance with Method 2 outlined above), oscillatory data is avoided by overbounding the envelope data $\bar{\ell}^2$ rather than raw data ℓ^2 (see Fig. 1).

The runs of Fig.4 are generated by fixing $q_0 = 0$, $q_1 = 1$ and varying the order m as m = 2,4,6. The envelope data $\bar{\ell}^2$ is depicted by the symbol '*'. It is seen that the overbounds for $m = 4$ and m = 6 are excellent, and in fact the latter nearly interpolates the data.

6. CONCLUSIONS

A systematic method, denoted as the LPSOF algorithm, has been developed for finding a minimum-phase transfer function of specified order whose magnitude "tightly" overbounds a specified nonparametric real-valued function of frequency. The main idea is to find a spectrally factorizable rational function which tightly overbounds the data "squared". This leads to a nonlinear constrained optimization problem which can be solved by a sequence of linear programming problems.

The original motivation behind the development of the LPSOF algorithm was to systematically replace the graphical overbounding method used in [4] for determining robust control weightings. However, the algorithm is generally useful for determining spectral factors from raw PSI] data and can be useful in such applications as deconvolution, disturbance identification, blind channel equalization, and estimation of noise coloring filters.

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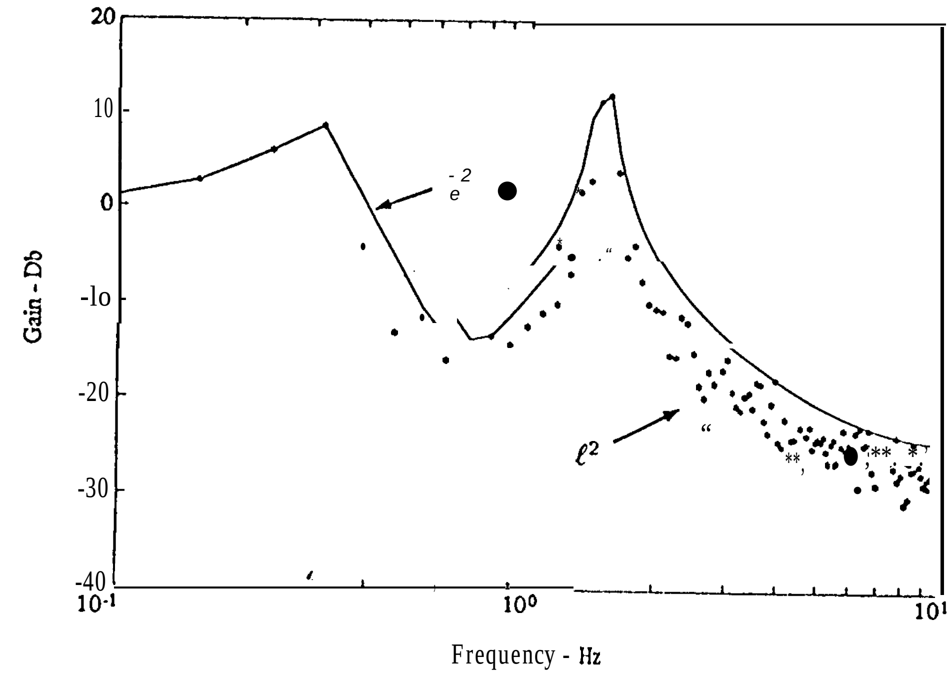

Figure 1. Raw additive uncertainty data ℓ^2 (denoted by '*') and envelope data $\bar{\ell}^2$ (denoted by '-') defined on 128 point grid.

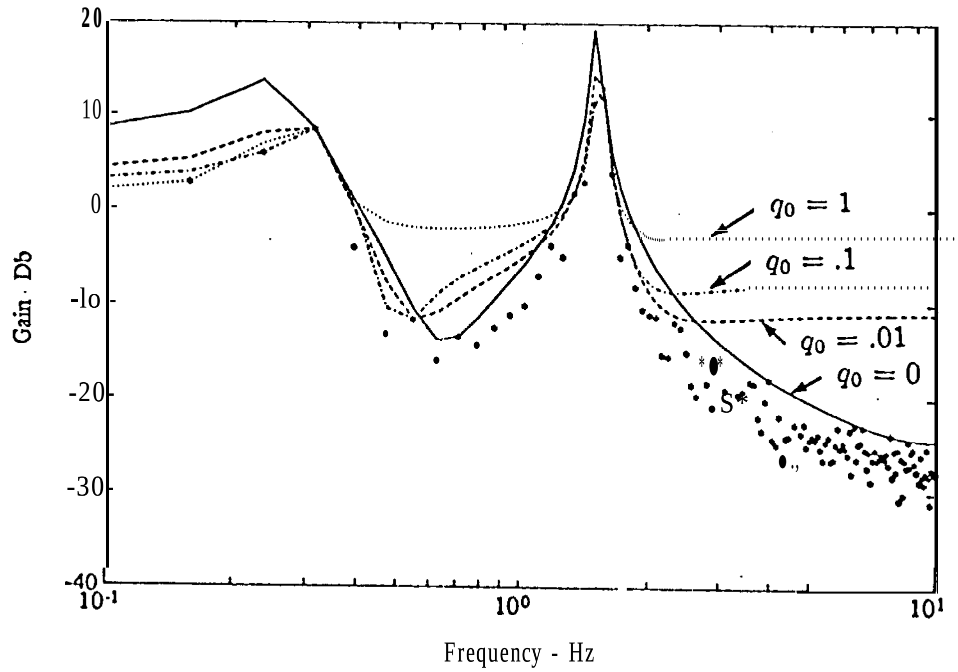

Figure 2. Spectrally factorizable overbounds β/α on *raw data* ℓ^2 (denoted by '*') using unmodified LPSOF algorithm ($\underline{\alpha} = \underline{\beta} = 0$, $q_1 = 1$, $m = 4$) obtained by varying weighting factor go = O, ,01, ,1, 1.

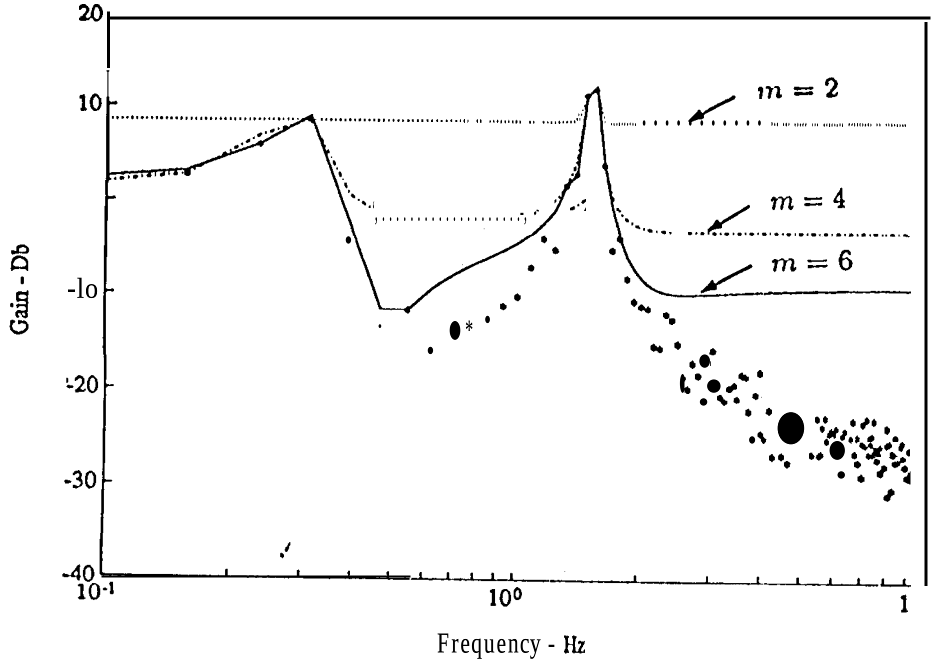

Figure 3. Spectrally factorizable **overbounds** β/α on *raw data* ℓ^2 (denoted by '*') using unmodified LPSOF algorithm ($\underline{\alpha} = \underline{\beta} = 0$, $q_0 = 1$, $q_1 = 1$) obtained by varying bound order m = 2, 4, 6.

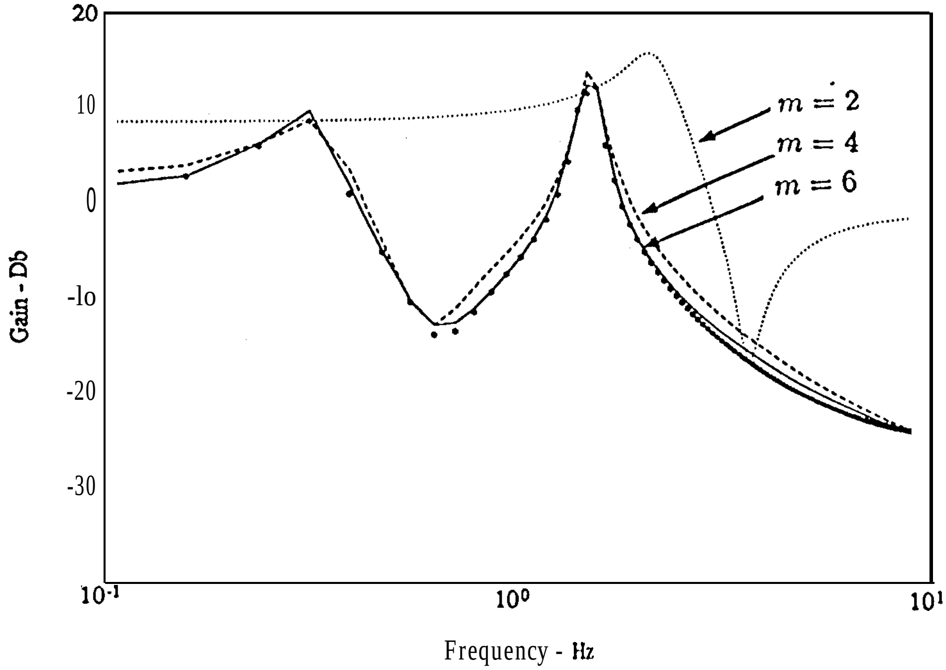

Figure 4. Spectrally factorizable **overbounds** β/α **on** *envelope* ***data*** $\bar{\ell}^2$ (denoted by '*') using unmodified LPSOF algorithm ($\underline{\alpha} = \beta =$ O, go = O, q_1 = 1) obtained by varying bound order m = 2, 4, 6.