Theory and Weighting Strategies of Mixed Sensitivity H^∞ Synthesis on a Class of Aerospace Applications [1]

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Abstract. This paper presents a vital design concept commonly used in the robust H^∞ control synthesis technique --- the *Mixed Sensitivity* H^∞ optimization. The underlying theory is also explained in a straightforward fashion. Several real world aerospace design problems are solved via this particular problem formulation. This simple approach provides control engineer a clean first cut of many complicated aerospace control design issues, e.g., stability, performance and robustness against frequency domain bounded unstructured uncertainty, etc. Only with this first cut result in hand, one can then move on to more advanced synthesis technique such as K_m-Synthesis to improve the system parametric robustness, if necessary.

Key Workds. Robust control; H^∞ weighting strategies; Aerospace applications

1. INTRODUCTION

For the past decade, robust control theory has made a "quantum leap" on the design of precision control systems in the presence of large level of uncertainty. The issues such as multivariable stability margin, multichannel loop-shaping, system robust stability and robust performance can be well formulated as one sound and complete mathematical problem, where one only needs to minimize the H^∞ norm of the input/output channels regardless it is a synthesis or analysis design problem. Figure 1 shows the "Canonical ***robust control*** pro *blcm*" setup,

In solving analysis problem, one can measure the "size" of the transfer function matrix seen by the uncertainty block(s) using the H^∞norm related tools to assess the multivariable stability margin. On the other hand, in solving synthesis problem, one can select a set of proper weighting functions to address a particular loop shape that ultimately takes care of the '(robustness" and "performance" design objectives in one mathematical framework. The tools that can be utilized to achieve the latter objective are, for example, H^2, H^∞optimization, and μ-synthesis procedure.

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However, unlike the simple nature of "analysis" problem, synthesizing a robust controller that stabilizes a plant (not necessarily a complicated one) with certain prescribed performance in the presence of all the anticipated disturbance, uncertainty, noise, etc. is absolutely a nontrivial task.

Mathematically, the Canonical ***Robust Control Probkm*** can be solved as follows:

Given a *multivariable* plant ***P(s)***, *find* ***a*** stabilizing controller F(s) such *that the closed loop transfer function* $T_{y_1u_1}$ *satisfies*

$$\mu(T_{y_1u_1}) = K_m^{-1}(T_{y_1u_1}(j\omega)) \leq 1 \quad (1)$$

where

$$K_m(T_{y_1u_1}) \stackrel{def}{=} \inf_{\Delta}\{\bar{\sigma}(\Delta) | det(I - T_{y_1u_1} A) = 0\} \quad (2)$$

with $A = diag(\Delta_1, \ldots . An)$.

From a robust control ***synthesis*** point of view, the problem is to find a stabilizing F(s) to "shape" the $\mu(T_{y_1u_1})$ function in the frequency domain. On the other hand, from a robust control analysis point of view, the problem is to compute the $K_m(T_{y_1u_1}$ function, or its bounds. In general, this robust control formulation is capable of handling multi-channel and multivariable control problem.

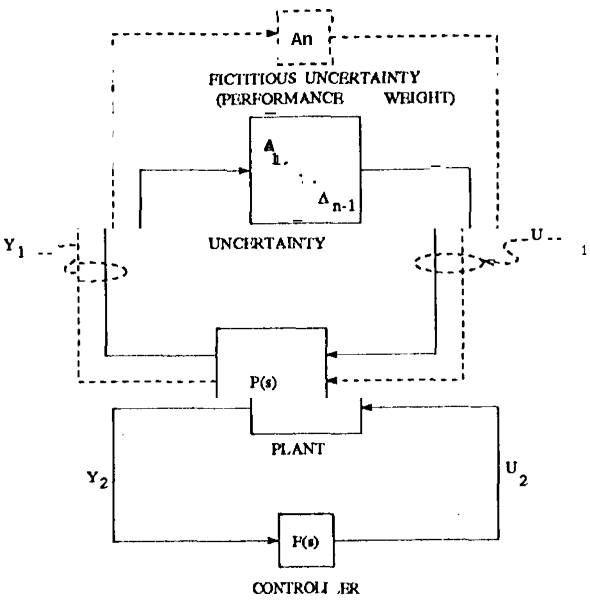

Figure 1: The Canonical Robust Control Problem

This paper catalogs a class of aerospace applications solved by H^∞ mixed sensitivity minimization, and describes their weighting strategies in detail. The applications covered here are 1) fighter flight control problem; 2) large space structure vibration suppression problem; 3) spacecraft attitude control problem, which are all related to the real-world design (not textbook problem) in practice. With the guidelines presented here and the examples of aerospace applications, robust control synthesis problem should no longer be a mystery to engineers or theorists that are new in this field.

2. H^∞ MIXED-SENSITIVITY APPROACH

To compute the K_m function in our Canonical *Robust Control Problem* setup is mathematically difficult (requires nonlinear programming), but the synthesis part of the problem can be closely approximated via the so-called Mixed-Sensitivity *Problem* setup (see Figure 2).

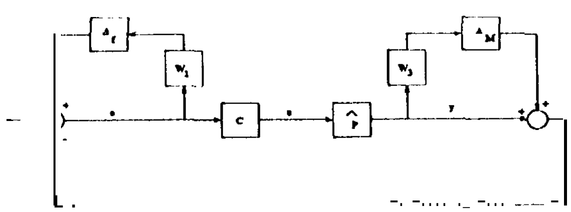

Figure 2: The Mixed-Sensitivity Problem.

First let's examing K_m function's upper bounds:

$$\frac{1}{K_m} \triangleq \mu(T_{yu}) \le \inf_{D\in\mathcal{D}} \|DT_{yu}D^{-1}\|_\infty \le \|T_{yu}\|_\infty \qquad (3)$$

where $\mathcal{D} := \{diag(d_1 I,\dots,d_n I)\,|\,d_i > 0\}$.

The inequality (3) implies that solving the "minimax" H^∞ optimization problem

$$\inf_{D\in\mathcal{D}} \sup_{s=j\omega} \bar\sigma\left[D(s)T_{yu}(s)D^{-1}(s)\right] \qquad (4)$$

minimizes an upper bound on the quantity $\mu(T_{yu})$. Doyle proves that for 3 or less complex singular-value-bounded uncertainty blocks, the first inequality become exact equality.

Singular value is also an upper bound of K_m^{-1}. It can be shown that in this H-mixed sensitivity setting, it is is only 3 db different from the true K_m function. This will be our main focus of solving the Canonical Robust Control problem. Let's start with the following inequality on the problem setup in Figure 2:

$$K_m^{-1}(T_{y_1u_1}) \le \bar\sigma(T_{y_1u_1}) = \bar\sigma\left(\begin{bmatrix} W_1S & -W_1S \\ W_3T & -W_3T \end{bmatrix}\right) \qquad (5)$$

where $S = (I+GF)^{-1}$ is the Sensitivity Function, $T = GF(I+GF)^{-1}$ is the *Complementary Sensitivity* Function, and clearly, $S + T = I$.

Take the singular value decomposition of the first column of T_{yu}

$$\begin{bmatrix} W_1S \\ W_3T \end{bmatrix} = U\Sigma V^T. \qquad (6)$$

and substitute back to T_{yu} we obtain the following upper-bound on K_m^{-1}:

$$\frac{1}{K_m} \le \bar\sigma\left(\begin{bmatrix} W_1S & -W_1S \\ W_3T & -W_3T \end{bmatrix}\right) =$$

$$\bar\sigma\left(U\left[\sqrt{2}\Sigma\right]\left[\frac{1}{\sqrt{2}}V^T \;\; -\frac{1}{\sqrt{2}}V^T\right]\right) \le \sqrt{2}\bar\sigma\left(\begin{bmatrix} W_1S \\ W_3T \end{bmatrix}\right) \qquad (7)$$

How about a *lower* bound ? We know if $D = diag(d_1, d_2)$, then

$$\frac{1}{K_m} = \inf_{d_1,d_2} \bar\sigma\begin{bmatrix} W_1S & -\frac{d_1}{d_2}W_1S \\ \frac{d_2}{d_1}W_3T & -W_3T \end{bmatrix}. \qquad (8)$$

Now, recall a fundamental singular value property:

The maximum singular value of any *matrix* is bounded below *by the maximum* singular value *of its submatrices.*

Thus, in particular one has the following **lemma:**

Lemma 1:

$$\inf_{\alpha\in(0,\infty)} \bar\sigma\left(\begin{bmatrix} X & \frac{1}{\alpha}X \\ \alpha Y & Y \end{bmatrix}\right) \ge \bar\sigma\left(\begin{bmatrix} X \\ Y \end{bmatrix}\right). \qquad (9)$$

Proof: If $\alpha \ge 1$,

$$\bar\sigma\left(\begin{bmatrix} X & \frac{1}{\alpha}X \\ \alpha Y & Y \end{bmatrix}\right) \ge \bar\sigma\left(\begin{bmatrix} X \\ \alpha Y \end{bmatrix}\right) = \lambda_{max}^{\frac{1}{2}}\left[X^*X + \alpha^2 Y^*Y\right], \qquad (10)$$

and

$$\lambda_{max}\left[X^*X + \alpha^2 Y^*Y\right] \geq \lambda_{max}\left[X^*X + Y^*Y\right] = \bar{\sigma}\left(\begin{bmatrix} X \\ Y \end{bmatrix}\right) \tag{11}$$

i e., the lemma holds. If $\alpha \leq 1$, the same logics yields

$$\bar{\sigma}\left(\begin{bmatrix} X & \frac{1}{\alpha}X \\ \alpha Y & Y \end{bmatrix}\right) \geq \lambda_{max}\left(\frac{1}{\alpha^2}X^*X + Y^*Y\right) \geq$$

$$\lambda_{max}\left(X^*X + Y^*Y\right) = \bar{\sigma}\left(\begin{bmatrix} X \\ Y \end{bmatrix}\right) \quad Q.E.D. \tag{12}$$

Using the above lemma (equation (9)-(1 2)), equation 7) finds its *lower* bound

$$K_m^{-1}(T_{yu}) \geq \bar{\sigma}\begin{bmatrix} W_1 S \\ W_3 T \end{bmatrix} \tag{13}$$

Combining the results of eq.(7) and eq.(13) with the robust stability requirement "$\sup K_m > 1$", one gets the following important relationship:

$$\boxed{\left\|\begin{bmatrix} W_1 S \\ W_3 T \end{bmatrix}\right\|_\infty \leq \sup_\omega K_m^{-1}(T_{yu}) \leq \sqrt{2}\left\|\begin{bmatrix} W_1 S \\ W_3 T \end{bmatrix}\right\|_\infty} \tag{14}$$

This relationship guarantees *that for the mixed* sensitivity *setup depicted* in Figure 2, *the 2-block H^∞* synthesis is *the same as the K_m synthesis (or $\mu(\cdot)$ synthesis) to within 3 db ($\sqrt{2}$) !*

The singular value upper bound in this inequality is known to be the so-called **H^∞ *Small-Gain* *Problem***, which by all means should be our first cut of the robust K_m synthesis problem. It replaces the complicated mathematical problem to an easy-to-solve H^∞ Mixed-Sensitivity problem. BY achieving $\|T_{y_1u_1}\|_\infty$ less than $\frac{1}{\sqrt{2}}$, one has achieved $K_m \approx 1$ - - the "real" robust performance.

Some important properties of the H^∞ controllers are listed below:

Property 1: The H^∞ optimal control cost function $T_{y_1u_1}$ is all-pass, i.e., $\bar{\sigma}[T_{y_1u_1}] = I$ for all $w \in R$. This property guarantees the exact loop shaping of H^∞ controller.

Property 2: An H^∞ "sub-optimal" controller has order equal to that of the augmented plant (n-state). An H^∞ optimal controller can be computed having at most (n - I)-states.

Property 3: In any *Weighted Mixed Sensitivity* problem formulation, the H^∞ controller always cancels the stable poles of the plant with its transmission zeros. For some plants with low-damped poles/zeros, this can potentially move the closed-loop poles into RHP and becomes unstable.

Property *4:* In the *Weighted Mixed Sensitivity* problem formulation, any unstable pole of the plant inside the specified control bandwidth will be shifted approximately to its $j\omega$-axis mirror image once the feedback loop is closed with an H^∞ (or H^2) controller (similar to the "Cheap" LQR control).

3. WEIGHTING STRATEGIES H"FORMULATION

A Small-Gain problem setup shown in Figure 3 has 3 most important signals (error, control energy, output) penalized around the control loop with the cost function

$$\min_{F(s)} \left\|\begin{bmatrix} W_1 S \\ W_2 F S \\ W_3 T \end{bmatrix}\right\|_\infty \leq 1 \tag{15}$$

It catches most control system design issues such as stabilit y, performance, and robustness in onc problem formulation. Most of all, it also provides a vital trade-off study among all these design issues. Namely, one can adjust each weighting function W_1, W_2 or W3 to come up a better design that suits his design requirement..

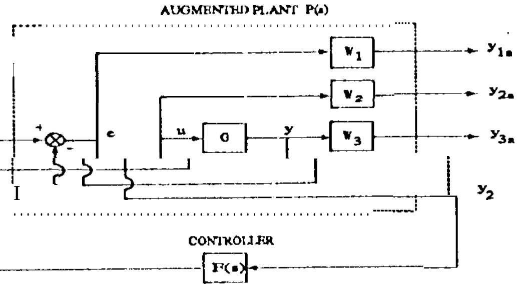

Figure 3: The Mixed-Sensitivit y Problem.

This setup yields the following open loop transfer functions:

$$P(s) = \left[\begin{array}{c|c} W_1 & -W_1 G \\ 0 & W_2 \\ 0 & W_3 G \\ \hline I & -G \end{array}\right] := \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]. \tag{16}$$

This will be the input to the software hinf.m or hinfopt.m Chiang and Safonvo (1988-1994) to compute an H 'controller.

Minimizing the H^∞ norm of the "plant" P(s) with proper weighting functions will result an all-pass closed loop cost function, which implies that one can get exact 100P shaping to within 3 db out of any of the two-block synthesis problems (ref. Section 2, Property 1):

$$P_1 = \begin{bmatrix} W_1 S \\ W_3 T \end{bmatrix}; \quad P_2 = \begin{bmatrix} W_1 S \\ W_2 F S \end{bmatrix}. \tag{17}$$

The following list summarizes some important rules (weighting strategies) associated with the H^∞ robust control synthesis, which is really a collection of facts from the fundamental H^∞ theory.

Rule # 1: For problem P_1, an necessary condition for an achievable H^∞ solution is

$$\bar{\sigma}(W_1^{-1}) + \bar{\sigma}(W_3^{-1}) > 1 \; \forall\omega \tag{18}$$

which means that the sum of the two weighting function singular values must be larger than 1 for all frequencies. This is simply due to the fundamental feedback "limitation" $S + T = I$. 'The weighting WI controls tracking error and disturbance rejection. The weighting W3 controls overall system bandwidth, roll-off rate and robustness against multiplicative uncertainty (see Figure 4). Together they form a desired loop shaping of the loop transfer function $L = GF$ along the frequency range of interest.

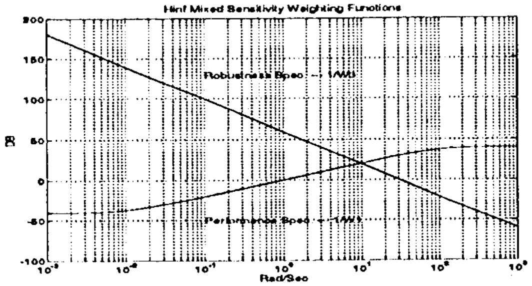

Figure 4: The standard weighting function W_1& W3 .

Rule # 2: For MIMO system, diagonal weighting function W_1 or W_3 forces the system to be "decoupled", which may or may not be a desirable thing to do depending on the physical problem (Ref. Section 4).

Rule # 3: The state-space H^∞ software currently coded in Chiang and Safonov (1988-1994) requires that the following conditions hold

$$rank(D_{12}) = dim(u_2) \leq dim(y_1) \qquad (19)$$

$$rank(D_{21}) = dim(y_2) \leq dim(u_1) \qquad (20)$$

i.e., D_{12} must be a "tall" matrix with full column rank, and D_{21} must be a '(fat" matrix with full row rank, Therefore, always including a non-trivial weighting W_2 ensures that D_{12} condition satisfies. For most engineering "tracking" control problems, D_{21} is always square, hence satisfying the D_{21} condition. However, there are cases like the one shown in Section 4, one must use P_2 formulation to solve a particular flight control problem without W_3 weighting.

Rule # 4: Always select stable and minimum phase weighting function, because

- Weighting functions are not stabilizable or detectable
- Poles of weighting function W_1 always becomes part of the poles of the H^∞ controller

Rule # 5: Preprocess the plant that has $j\omega$-axis zeros or poles. Otherwise, it can cause the H^∞ algorithms to fail. This is due to the fact that when H^∞ cost function approaching "optimal", the overall closed loop system will have an irrational transfer function with point discontinuities on the $j\omega$-axis at the offending $j\omega$-axis poles and zeros of the plant (Ref. Safonov, 1986 for details). Solutions have been developed to deal with such situations:

- For plant has $j\omega$-axis poles, a simple bilinear pole-shifting transform

$$s = \frac{\tilde{s} + p_1}{\frac{\tilde{s}}{p_2} + 1} \qquad (21)$$

can map the $j\omega$-axis onto a RHP circle Γ_2, while preserving the H^∞ norm of the problem. After solving the problem on the circle (instead of on $j\omega$-axis), simply apply the inverse bilinear transform to the H^∞ controller to go back to the original domain (see Figure 5, Ref. Chiang and Safonov, 1992 for more details).

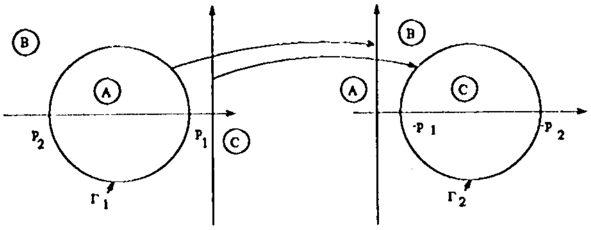

Figure 5: The bilinear pole-shifting transform (from $s-plane$ to $\tilde{s}-plane$).

- For plant has $j\omega$-axis zeros (any strictly proper plant), attaching improper weighting function W3

$$W_3(s) = C(Is+A)^{-1}B + D + \alpha_n s^n + \ldots + \alpha_1 s + \alpha_0 \qquad (22)$$

can not only penalize plant roll-off rate against unstructured uncertainty but also keep the size of the augmented plant $P(s)$ unchanged. The state-space form of this special kirrd of plant augmentation has been implemented in Chiang and Safonov (1988-1994) (augtf.m and augss.m) based on the theory of state-space resolvant.

Rule # 6: Use some engineering judgement before demanding H^∞ software to find a controller for you. For example, one can not suppress sensitivity function to be less than one at vicinity of RHP transmission zeros. This is one fundamental feedback control limitation (not H^∞).

Rule # 7: Balance the augment plant in equation (16) for a better numerical condition so that the Riccati equation solver can be well-posed.

This set of rules must be kept irr mind in every H^∞ control designer.

4. DESIGN EXAMPLES

The following aerospace design examples utilize this H^∞ Mired-Sensitivity problem formulation described in equation (17) to achieve their requirements.

Example # 1: Flight Control Bank Turn (Thompson and Chiang, 1988). An interesting flight control problem that requires coordination between bank angle ϕ and stability yaw rate r_s is solved using the standard H 'mixed sensitivity problem formulation, where $W_1(s) = diag[\frac{0.02s+1}{s+0.01}, 0.01]$, $W_2(s) = 2 * diag[\frac{0.1s+1}{0.01s+1}, \frac{0.1s+1}{0.01s+1}]$, and no W_3

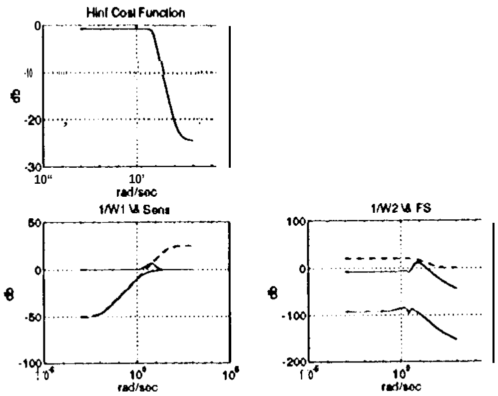

Figure 6: H^{∞}mixed sensitivity design (bank turn).

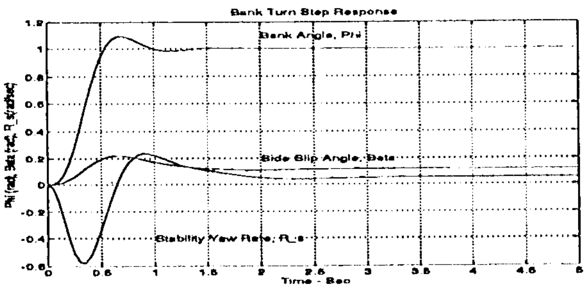

Figure 7: Bank turn step response.

weighting! If one uses diagonal W_1 and W_3 weighting. on ϕ and r_s, the side-slip angle β will diverge quickly. Because in any airplane bank turn situation, $r_s = g tan\phi / V_T$, decoupling ϕ from r_s is against the physics law. However, in other situation like the Himat flight control problem, standard W_1 and W_3 are necessary weightings to decouple the θ and α variables for a Direct Lift flight control design (Safonov and Chiang, 19S8). See Figures 6 and 7 for H^{∞}bank turn design.

Example # 2: Structure Vibration Suppression (Bayard and Chiang, 1993) An integrated ID and robust control design methodology (MACSYN) has been developed in JPL (Bayard *et al.*, 1994) In this design, H^{∞}mixed sensitivity approach was again used effectively to remove the most critical unwanted vibration modes. Weighting W_2 is an overbound on all the identification and modeling errors. Weighting *WI* is chosen to home in just the first 3 bending modes. It is a 6-state modal model truncated from the full 100-state plant seen from the disturbance actuator to the accelerometer. Figures 8 and 9 list the outstanding performance of this approach.

Example # 3: Rigid Body Attitude Control (Chiang *et al.*, 1993). Controlling rigid body dynamics is a very common industrial task. From EM actuator, spacecraft dynamics to any rotating object that needs to be precisely controlled, we have a double integrator plant: $\frac{1}{Js^2}$, where J is the polar moment inertia. A spacecraft rigid body with J = 5700 has

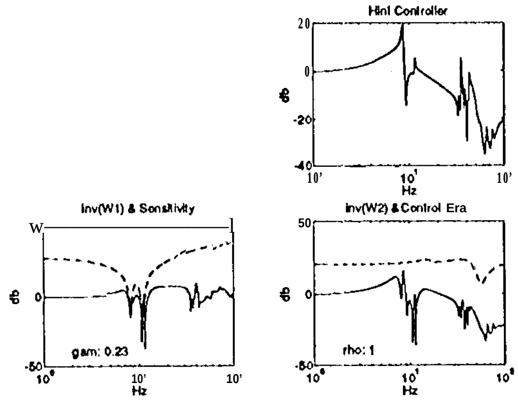

Figure 8: H^{∞}mixed sensitivity design (vibration suppression).

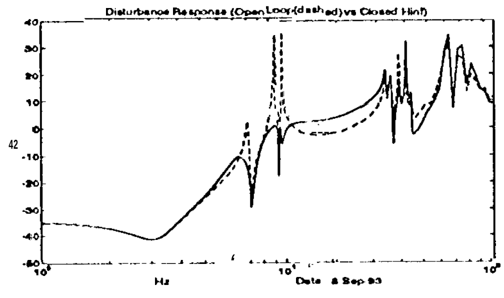

Figure 9: Disturbance rejection (open vs. closed loop).

been controlled using H^∞ mixed sensitivity formulation. Here we have $W3 = \frac{s^2}{100}$ and

$$W_1 = \frac{\beta(\alpha s^2 + 2\zeta_1 \omega_c \sqrt{\alpha} s + \omega_c^2)}{(\beta s^2 + 2\zeta_2 \omega_c \sqrt{\beta} s + \omega_c^2)} \qquad (23)$$

where $\beta = 100$ is the DC gain that controls the disturbance rejection, cs = 1/1.5 is the high frequency gain that controls the peak overshoot response, $\omega_c = 3$ is the sensitivity cross-over frequency, $\zeta_1 = \zeta_2 = 0.7$ are the damping of the poles **and zeros. Additional attention** needs to pay for the $j\omega$-**axis** plant poles (Rule # 5 !). We simply shift the plant "A" matrix by 0.1 to the right ($A_g = A_g + 0.1$), then shift back the final H^∞ controller to the left by the same amount ($A_{cp} = A_{cp} - 0.1$). **This is** equivalent to a bilinear mapping with circle coefficients $p_1 = -0.1$ **and** $p_2 = -\infty$. This kind of mapping guarantees that we have a strictly proper controller that never amplifies sensor noise at high frequencies. Figure 10 shows the overall design.

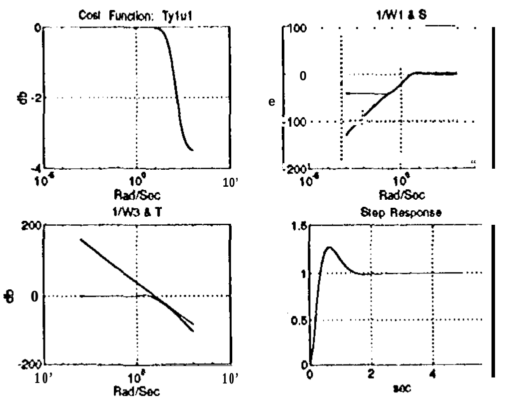

Figure 10: H^∞ mixed sensitivity design (double integrator plant).

5. CONCLUSION

H^∞ Mixed Sensitivity problem formulation provides control designer the first clean cut of any complicated control problem. Mathematically, it has the advantage of bypassing the difficult K_m computation. From control design viewpoint, it provides direct design knobs on the loop transfer function, which essentially solves the fundamental feedback issues like stability, tracking performance, disturbance rejection, and robustness against unstructured uncertainty. This paper documents the basic theory, weighting strategies and three nontrivial aerospace design examples to show the merits of this approach. More advanced technique such as Km-Synthesis (Chiang and Safonov, 1992; Safonov and Chiang, 1993) can then be invoked to focus on the parametric robustness of the problem, after the H^∞ mixed sensitivity problem is solved. A similar K_m tutorial paper like the one presented here will be published elsewhere.

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