PLUTO'S HELIOCENTRIC ORBIT

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Abstract

We review the current state of knowledge regarding Pluto's heliocentric orbital motion. Pluto's orbit is unusually eccentric and inclined to the ecliptic, and overlaps the orbit of Neptune. Consequently, Pluto suffers planetary perturbations to a greater degree than any of the other planets, and its orbit evolves significantly over long times. The current uncertainty in Pluto's orbital parameters and its implications for the long term dynamical evolution of Pluto is reviewed. Numerical integrations of increasingly long times into the future indicate that Pluto exists in a dynamical niche - consisting of several resonances which protects it from close encounters with the giant planets, but which may also introduce a chaotic component in its motion. The extent and character of this dynamical niche is described. The emplacement of Pluto in this niche requires some dissipative mechanism in the early history of the Solar System. We discuss two origin scenarios for this unusual orbit, one invoking a catastrophic giant impact, the other a relatively slow evolution of the orbit.

1. Introduction

The heliocentric motion of Pluto is of great interest for several reasons. It is intrinsically interesting because Pluto's orbit departs very significantly in character from the usual well-separated, near-circular and co-planar orbits of the major planets of the Solar System. During one complete revolution about the Sun (in a period of 248 years at a mean distance of about 4(I AU), Pluto's heliocentric distance changes by almost 20 AU from perihelion to aphelion, and it makes excursions of 8 AU above and 13 AU below the plane of the ecliptic (see Figure 1); for approximately two decades in every orbital period, Pluto is closer to the Sun than Neptune. Furthermore, Pluto is accompanied in its orbit about the Sun by a large satellite, Charon; the large mass ratio Of Charon to Pluto makes this truly a binary planet. The origin and stability of this binary planet in a very peculiar orbit in the outer reaches of the planetary system is a fascinating question in Solar System dynamics and may 1101(1 clues to planet formation processes in the outer Solar System. Pluto's orbital history is also of importance for the geophysical and climate evolution of this system (cf. chapters in *BULK PROPERTIES* and *ATMOSPHERES*).

It was recognized early on that Pluto's[1] orbital period is exactly 3/2 that Of Neptune. Owing to this orbital resonance and Pluto's large eccentricity and inclination, the usual analytical methods of celestial mechani'csave been of limited use in determining the long term motion of Pluto under the influence of])('J'till'l)iifi[)Ils from the giant planets. Therefore, most studies of Pluto's orbital dynamics have involved numerical integrations of increasingly long times. The enormous increase in computing speed facilitated by digital computers and faster numerical integration algorithms in recent years now allows the exploration of planetary dynamics over billion year timescales with relative ease. We now know that Pluto's long term motion exhibits a rich variety of dynamical phenomena: the strong mean motion resonance with Neptune, several resonances and near-resonances with the secular motions of the giant planets, as well as evidence of deterministic chaos. The

[1] 111 the interest of brevity, wc will refer to the heliocentric motion of the center-of-mass of the Pluto-Charon binary as simply that of 'Pluto'.

latter is especially curious, because numerical evidence also suggests that over billion year timescales Pluto is secure from macroscopically large changes in its orbital parameters. This complicated dynamics has motivated two new scenarios for the origin of this orbit; it appears plausible that Pluto formed in a more ordinary near-circular, co-planar orbit beyond Neptune and was transported to its current peculiar orbit by dynamical processes in the early history of the Solar System. We note that these theories are a striking departure from the early speculations of Pluto being an escaped Neptunian satellite.

This chapter reviews the current state of knowledge about the orbit of Pluto and is organized as follows. Section 2 describes the history of Pluto's orbit determination and discusses the quality of the present ephemerides and prospects for improvement in the future. Section 3 describes Pluto's orbital evolution on long time scales. In Section 4 we discuss the mechanisms that determine Pluto's orbital stability. In Section 5 we review two most plausible scenarios for the origin of Pluto's orbit. Section 6 summarizes the chapter and indicates avenues for future studies.

2. Current Orbit

Although Pluto was discovered in 1930, there exist prediscovery photographs which provide its positions back to 1914. Thus Pluto has been observed for nearly 80 years, or 1/3 of its orbit period. The following osculating, heliocentric elements in the J2000 coordinate system are taken from the most recent planetary and lunar ephemeris, DE 245, by Standish, Newhall, and Williams (1993) and use a Sun/Pluto mass ratio of 135,000,000 (Beletic *et al.* 1989). They are based on approximately 900 astrometric positions observed over nearly eight decades. In addition to the six Keplerian elements (semimajor axis a, eccentricity e, inclination i, node Ω, argument of perihelion ω, and mean anomaly M), some auxiliary quantities (mean motion n, orbital period, perihelion distance q, and aphelion distance Q) are also listed.

Pluto's orbital elements

epoch	MJD 40400.0
a [AU]	39.77445 ± 41
e	0.2533182 ± 55
i [deg]	17.13487 ± 3
Ω [deg]	110.2?8631 ± 19
ω [deg]	112.98240 ± 130
M [deg]	331.37659 ± 130
n [deg/day]	0.00392914 ± 6
period [y]	250.8502 ± 40
q [AU]	29.69886 ± 11
Q [AU]	49.85004 ± 73

These elements are affected by short-period planetary perturbations. If those effects are removed, the average (over a few centuries) orbital period is 248 yr.

Orbit determination for observation times less than an orbit period give best accuracies in the observed arc of the orbit and degraded accuracies elsewhere. The difficulties stemming from an incomplete orbit are complicated further by systematic star catalog errors. Pluto's orbit suffers from nonuniform accuracy: the semimajor axis is less well known than the perihelion distance (perihelion passage occurred in 1989 and is in the observed arc), the perihelion direction and mean anomaly are coarser than the other angular elements (but the mean longitude, $\lambda = \omega + \Omega + M$, is known nearly an order of magnitude better, 0.00015°), and the error ellipse for the orbit pole is elongated by 2 to 1.

The uneven orbit accuracy shows up in predictions of Pluto's future position. Predictions only a decade beyond the last observation are noticeably in error (Seidelmann et al. 1980, Standish 1993). At present the least well known coordinate is the radial distance with an uncertainty which exceeds 10,000 km. A future spacecraft mission to Pluto would benefit from high ephemeris accuracies. Lower accuracies cause pointing uncertainties. To maintain the highest accuracy in the future, it is necessary to make positional observations and to regularly update the orbit.

3. Long term evolution

Like the innermost planet Mercury, the orbit of distant Pluto is distinguished from the other planets by the size of its eccentricity and inclination. Figure 1 shows the orbits of the five outer planets. The extent of both radial and out-of-ecliptic-plane excursions far exceeds those of all other major planets. Pluto's perihelion distance is smaller than Neptune's mean distance - indeed its present perihelion (29.7 AU) is slightly smaller than Neptune's (29.8 AU, both with short-period variations removed). The question naturally arises whether close approaches between Pluto and Neptune prevent orbital stability. The size of the eccentricity and inclination of Pluto's orbit and its crossing perihelion do not make it an attractive subject for studies by analytical perturbation theory. Consequently, numerical integrations have dominated the studies of Pluto's orbit evolution and the speed of available computers has limited the length of those integrations. Over the past three decades the succession of integrations with longer and longer times is testimony to the improvement in computer speed and innovative numerical integration algorithms. A summary of these numerical integrations follows.

Numerical Integrations of Pluto

Authors	Pub Date	Span	Comp Time
Cohen & Hubbard	1965	120 kyr	3 d
Cohen, Hubbard & Oesterwinter	1967	300 kyr	
Williams & Benson	1971	4.5 Myr	1 hr
Cohen, Hubbard & Oesterwinter	1973	1 Myr	
Kinoshita & Nakai	1984	5 Myr	4 hr
Milani *et al.*	1986	9.3 Myr	
Applegate *et al.*	1986	217 Myr	14 d
Sussman & Wisdom	1988	845 Myr	
Richardson & Walker	1988	1 Myr	65 d
Milani, Nobili & Carpino	1989	100 Myr	
Quinn, Tremaine & Duncan	1991	3 Myr	65 d
Wisdom & Holman	1991	1 Byr	14 d
Sussman & Wisdom	1992	100 Myr	40 d

Planetary perturbations can be divided into short and long period effects. The short-period perturbations depend on the positions of the bodies in their orbits, *i.e.* on the recall anomalies or mean longitudes. The longer period effects, commonly called secular perturbations, include the secular motions of nodes and perihelion directions and long-period variations in nodes, perihelion directions, eccentricities, and inclinations. Pluto exhibits resonances with both types of perturbations.

3:2 Resonance

Cohen & Hubbard (1965) integrated the five outer planets for 120,000 yr. They discovered that the orbit of Pluto is locked in a 3:2 mean motion resonance (commensurability) with Neptune. During every five centuries, Pluto makes two revolutions and Neptune three, and the two planets pass one another once. After five centuries the geometric pattern nearly repeats (see Figure 2). A resonance argument, ϕ, can be defined using the mean longitudes of Pluto and Neptune, λ and λ_N respectively, and the longitude of Pluto's perihelion, ϖ,

$$\phi = 3\lambda - 2\lambda_N - \varpi \tag{1}$$

Cohen & Hubbard found that this argument librates about 180° with an amplitude of 76° and a period of 19,700 yr. The importance of the libration about 180° can be seen by writing the resonance argument as

$$\phi = M - 2(\lambda_N - \lambda), \tag{2}$$

where $M = \lambda - \varpi$ is Pluto's mean anomaly. For Pluto to be at perihelion ($M = 0$) while passing Neptune ($\lambda \approx \lambda_N$), the resonance argument, ϕ, would need to approach zero. Thus the libration of ϕ about 180° prohibits very close approaches and causes Pluto's conjunctions with Neptune (i.e. the configuration when the two planets share the same heliocentric longitude) to be closer to Pluto's aphelion than perihelion. Another way to understand the resonance protection is to note that the libration of ϕ about 180° means that at perihelion ($M = 0$), Pluto's mean longitude is near 90° away from Neptune's longitude, thereby avoiding conjunctions of the two planets when Pluto crosses the orbit of Neptune. This is shown in Figure 2 in a coordinate system rotating with Neptune's mean rate.

Cohen and Hubbard showed that over five century cycles the distance between Pluto and Neptune has three minima, the smallest of them (18 AU) occurs when the planets have similar heliocentric longitudes and Pluto is near aphelion. The other two minima occur closer to Pluto's perihelion (at the small loops in Figure 2), but the longitudes of the two planets are very different and the distances are larger. Figure 3 illustrates how the distance between Pluto and Neptune changes during the 20 kyr libration. It is interesting to note that Pluto makes closer (and more frequent) approaches to Uranus than to Neptune (see Figure 4). However, the Pluto-Uranus distance varies so rapidly in successive close approaches that the Uranian short-period perturbations are periodic over only a few thousand years and do not accumulate significantly over longer time scales.

Subsequent 3(KI kyr and 1 Myr integrations (Cohen, Hubbard & Oesterwinter 1967, 1973) revised the libration amplitude of ϕ to 80° and slightly shortened the libration period. Even longer numerical integrations since these original studies have confirmed the 3:2 resonance libration and the protection it provides against dose approaches with Neptune (see Figure 5).

ω Libration

The major planets exhibit sizable '(see.ula]" variations on time scales from 46 kyr to 2 Myr. These variations are not associated with the fast time scale of the orbit periods, but with the much slower precession of the perihelia and nodes. During Cohen and Hubbard's original 120 kyr integration the argument of Pluto's perihelion, ω, moved only 1.4°. Because the present perihelion and aphelion are 16° out of the plane of the ecliptic, the possibility remained that the 3:2 libration would not survive for times comparable to either the circulation of the perihelion or the secular perturbations. Even if the 3:2 resonance remained locked in libration, the possibility existed that the closest approach distance would be reduced when the encounter point got closer to the ecliptic plane. But commenting on the very slow argument of perihelion motion during 120 kyr, Brouwer

(1996b) suggested another possibility: ω might librate instead of circulating.

A longer integration of 4.5 Myr was undertaken by Williams and Benson (19'(1). Numerical averaging techniques were used to keep the computer usage modest. It was found that the argument of perihelion librated around 900 with a 4 Myr period and a 24° amplitude. During the integration the minimum distance between the two planets was found to be 1 7 AU. The libration of ω keeps the closest approach point out of the plane of Neptune's orbit making the minimum distance larger. Both librations tend to make the cumulative perturbations smaller in magnitude than they would otherwise be.

The 4 Myr periodicity of Pluto's argument of perihelion is accompanied by 4 Myr osc.ills.tiolls of the eccentricity and inclination. The inclination varies 2° peak-to-peak and the eccentricity varies by 0.05. The extrema of these e and i variations (one a maximum and the other a minimum) occur when ω = 90') In other words, when the aphelion contracts toward the Sun its latitude increases. This helps to increase the minimum Neptune-Pluto distant.c.

For orbits of modest inclination, planetary perturbations normally dr ve the perihelion precession prograde. The perturbations by Jupiter, Saturn, and Uranus follow this pattern. The 3:2 commensurability with Neptune, however, contributes a retrograde precession. As a result, the net rate of precession of Pluto's argument of perihelion is always small and averages zero (the longitude of perihelion rate is negative and on average is equal to the average node rate). An early attempt (Hori and Giacaglia 1968) to analytically compute Pluto's orbit evolution based 011 three-body theory (Sun, Neptune, Pluto) failed to find the ω libration, but subsequent work based on periodic orbits and semianalytic techniques using multiple perturbing planets (Nacozy and Diehl 1 974, 1 978ab) did confirm the ω libration. The first direct numerical integration (no averaging techniques or semianalytic treatment) with a long enough span to confirm the ω libration was that of Kinoshita and Nakai (1984) who integrated the five outer planets for a timespan of 5 Myr. They found that the ω libration had a 3.8 Myr period and an amplitude of 23°.

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Williams and Benson (1971) had suggested that Pluto might exhibit two other "secular" resonances involving Pluto and Neptune's node and perihelion precession rates and Pluto's argument of perihelion libration, but their 4.5 Myr integration was not long enough to confirm this possibility. Since that work, much longer integrations have been performed (100 Myr 1 Byr, see below). Long period variations of Pluto's orbital elements were reported by Applegate *et al.* (1986) in their 210 Myr integration. Some of these were subsequently identified by Milani *et al.* (1989) with the secular resonances proposed by Williams & Benson.

Applegate *et al.* (1986) used a special purpose computer - the *Digital Orrery* - to numerically integrate the orbits of the outer planets (the 4 giant planets plus a zero-mass test particle representing Plute) for :I100 Myr. This was a leap by a factor of $\sim$ 40 over the longest integrations till that time. Pluto's orbit was determined to be stable and the previously known librations of ϕ and ω were preserved over this time span. Some interesting new features also emerged. The ω libration was found to be modulated with a 34.4 Myr period, its amplitude varying between $\sim 18°$ and $\sim 28°$ (Figure 6). Because Pluto's e and i are strongly coupled through the ω libration, both these orbital elements also exhibit significant validations. The 3.8 Myr oscillations of i were found to be strongly modulated by the 34 Myr period, and there were indications of even longer period variations. $h = e \sin \omega$ was reported to exhibit a strong modulation with a 27 Myr period, as well as a 137 Myr period.

Sussman & Wisdom (1988) extended the Digital Orrery integration to a time span of 845 Myr. This integration confirmed all the above features in Pluto's motion. They also reported a 150 Myr period in the variations of Pluto's inclination and indications of an even longer period of a])]~lc)xill-lately 600 Myr. In addition, they found evidence of deterministic chaos (see below and Figure 7).

An independent 100 Myr numerical integration of the outer planets was performed by the LONGSTOP project (see Nobili 1988 for a review of this work). In a thorough paper on the analysis of their numerical solution for Pluto, Milani *et al.* (1989) discovered that there was a 1:1 commensurability between the libration period of ω and the circulation period of $(\Omega_N - \Omega)$. They refer to this as a "super-resonance", and identified the 34.5 Myr modulations with the libration period of the super-resonance. This resonance has a geometrical consequence when considering the inclination of Pluto's orbit: the 3.8 Myr variation in the inclination with respect to the invariable plane is opposed to the inclination variations of Neptune's orbit (actually the coefficient associated with the principal frequency) so that Pluto's inclination with respect to Neptune's orbit plane varies by only 1°.

Milani *et al.* also attempted to determine the origin of the longer period perturbations and the signature of chaos detected in the Orrery calculations. They identified yet another "super-resonance": the difference of the longitudes of perihelion of Pluto and Neptune, $(\varpi - \varpi_N)$, circulates with a period of 1.267 Myr,:ry close to 1/3 of the 3.8 Myr circulation period of $(\Omega_N - \Omega)$. (Because of the first super-resonance, this guarantees a commensurability with the ω libration also.) In the LONGSTOP integration, the combination of angles, $(\varpi - \varpi_N)\, 3(\Omega_N - \Omega)$, was close to but not locked in resonance; however, in the Orrery integration, the average rate of this angle was indistinguishable from zero within numerical resolution. Milani *et al.* have suggested that the origin of the chaos in the Orrery calculation could be this super-resonance. Whether Pluto is locked in this resonance remains an outstanding question.

Comparison of integrations

The accuracy of Pluto's orbital elements has improved with time and our knowledge of the outer planet masses benefited greatly from the Voyager flybys. The influence of mass correction and orbit uncertainties on the long-term orbit evolution will be considered here. The 210 Myr integration by Applegate *et al.* (1986) and the 100 Myr integration of Milani *et al.* (1989) are the longest that have been analyzed for their fundamental frequencies and amplitudes and will be used here.

Most of the integrations have used a Neptune mass which is 0.51% larger than the Voyager value; the value used by Milani *et al.* is 0.10% smaller than Voyager's. To a good approximation the 3:2 libration frequency scales as the square root of Neptune's mass. When corrected to the Voyager value, the libration periods from both Milani *et al.* and Applegate *et al.* are within 6 yr of 19,912 yr.

The 3:2 libration amplitude is very sensitive to the measured mean motion. The libration causes Pluto's mean motion to oscillate by 0.6% about the mean value imposed by the derivative of the resonance argument. Consequently, relative mean motion uncertainty is amplified by two orders-of-magnitude when determining libration amplitude uncertainty. For the mean motion uncertainty given in Section 2, the amplitude uncertainty is 0.1°. The two papers give peak, rather than mean, libration amplitude, and they differ by 2° (84° for Milani *et al.* and 86° for Applegate *et al.*). As the libration center oscillates by 3° about 180° (Williams & Benson 1971, Milani *et al.* 1989), the mean libration amplitude appears to be about 82°, but is somewhat uncertain.

The libration period for the argument of perihelion, ω, is near 3.8 Myr. Applegate *et al.* give 3.796 Myr and Milani *et al.* find 3.783 Myr. The 0.3% difference is plausibly due to mass differences since the libration frequency depends upon the square root of a linear combination of all of the outer planet masses, but the coefficients of the linear combination are not known. Assuming that the libration frequency is dominated by Neptune's mass gives the right size correction, but the wrong sign.

Because the ω libration is resonant with the difference in the nodes of Pluto and Neptune, $(\Omega_N - \Omega)$, consideration of the sensitivity of the node precession rate to outer planet masses should permit adjustment. This can only be done approximately, but the approximation from Laplace-Lagrange theory is better known than that for the ω libration. For Pluto's average node rate, both integrations correct to - 0.3502″/yr. The average longitude of perihelion rate will be the same (owing to the ω libration). For

Neptune's average node rate, the discrepancy between the two integrations is made worse by adding a mass correction (- 0.6921″/yr for Applegate *et al.* and - 0.6930″/yr for Milani *et al.*). Mass corrected, the periods for a full cycle of $(\Omega_N - \Omega)$ are 3.791 Myr and 3.781 Myr, and these should match the ω libration periods.

Neptune's average longitude of perihelion rate is also of interest. Here the agreement is good and a mass corrected value is 0.6730″/yr. The difference of thelongitudes of perihelion Of Pluto and Neptune, $(\varpi - \varpi_N)$, completes a cycle in] .267 Myr, very close to 1/3 of the periods of the ω libration and the circulation] of $(\Omega_N - \Omega)$. For the combination of angles, $3(\Omega - \Omega_N) - (\varpi_N - \varpi)$, the mass corr ected rate for Applegate *et al.* is 0.0024″/yr, about half the Milani *et al.* value of 0.0051″/yr. We cannot predict whether the mass corrected value of Applegate *et al.* would be resonant.

The inability to correct to the same rates in the two integrations is due to some combination of the approximations used for the mass sensitivities, the resona nces, different initial conditions for the planets, and modeling. The planetary orbits have now been fit using the improved planetary 111.asses and could serve as starting conditions for long integrati ons. Rather than trying to reconcile the consequences of different masses and initial conditions of past integrations, the time seems ripe to perform new integrations which eliminate these uncertainties.

The average ω rate is zero, but the libration causes the rate to vary with time. The average ω amplitude from the two long integrations is between 21° and 22°. There should be a small influence of the difference in the Neptune II iass, approximately ().1°, on the amplitude.

4. Dynamical stability and chaos

The dominant perturbations on Pluto's motion arise from Neptune. Because Pluto's mass is $\sim 10^{-4}$ that of Neptu ne, the sim plest n iodel for analyzing these perturbations is to consider the planar restricted three body problem consisting of the Sun and Neptune as the massive primaries in circular orbits about their center of mass, and Pluto as a massless test particle. This model has the important advantage that tjICStI'IIC.tiII'C(Jft]IC phase space can be visualized in a 2-D surface-of-section. In such a picture, quasiperiodic (i.e. secularly stable) motion appears as points that lie on a smooth, closed curve, while chaotic (or secularly unstable) motion appears as points that fill up a 2D region (see Henon 1983 for further details). Figure 8 shows the structure of the phase space in the vicinity of the 3:2 Neptune-Pluto resonance in terms of canonical resonance variables. In obtaining this Sill'faC.C-Of-SCC.tiOl"l, the value of the Jacobi integral was set equal to that for the observed Pluto but with its inclination suppressed. It is obvious from this that stable librations of the resonance angle are possible only in a narrow region of the phase space. The approximate half-width of this stable resonance region in terms of semimajor axis is only $\Delta a \approx 0.5\ AU$. Librations with amplitude greater than $\sim 100°$ are chaotically unstable on very short tilllcsc.ales, $O(10^4)yr$. The observed libration amplitude of Pluto is 82°, and is inside the stable region. Thus, within this approximate model, Pluto's motion is bounded and stable for all time.

The question naturally arises whether the motion remains stable in the realistic case, in which Pluto has a non-zero inclination]], Neptune's orbit is not on a fixed circle but has a small eccentricity and inclination, and the perturbations of the other planets are also included. It can be argued that the third degree of freedom (Pluto's inclination) by itself will not make the orbit unstable. However, taking account of the non-circular orbit of Neptune, and the perturbations of the other planets introduces new dynamical features whose effects on Pluto's long term orbital stability are more difficult to analyze. The most significant of these is the ω libration described in the previous section. The high inclination of Pluto's orbit together with the ω libration helps to keep Pluto's perihelion out of the ecliptic plane, and therefore helps reduce the magnitude of the planetary perturbations. Other weak resonances ("super-resonances" in Milani *et al.* 1989) that have been identified in the long term numerical integrations of the outer planets also have the effect of increasing the closest approach distance between Pluto and Neptune.

On the other hand, it is well established that resonance regions are accompanied by chaotic zones in phase space. The relevant question, therefore, is whether the dynamical protection mechanisms remain robust for a sufficiently wide range of initial conditions and parameters that encompass those of the actual Solar System. We discuss this below.

Lyapunov exponent

Chaotic solutions of a dynamical system are characterized by an extreme sensitivity to initial conditions which is most directly measured by its maximal Lyapunov exponent, Γ. This is the quantity that measures the rate of exponential divergence of two trajectories in phase space that initially are arbitrarily close to each other:

$$\Gamma = \lim_{d(0)\to 0} \lim_{t\to\infty} \frac{\ln\left(d(t)/d(0)\right)}{t}. \tag{3}$$

Here $d(0)$ is the initial separation in phase space and $d(t)$ the separation at time t. In a regular region of phase space, Γ is zero; in a chaotic region it is finite and positive. The associated timescale for chaotic divergence of orbits is $T_L = \Gamma^{-1}$. In practice, in numerical experiments one determines the so-called finite-time maximal Lyapunov exponent,

$$\gamma = \frac{\ln\left(d(t)/d(0)\right)}{t}, \tag{4}$$

where $d(0)$ is small but non-zero. Then, at increasingly large t, γ asymptotically approaches Γ.

The Lyapunov exponent, γ, for Pluto's motion has now been determined in several long numerical integrations. In the first of these (Sussman & Wisdom 1988), the orbits of the four massive outer planets and a massless "Pluto" were integrated for a period of 845 Myr using a special purpose computer (the *Digital Orrery*) and the multi-step Stormer integrator. The maximal Lyapunov exponent was found to be $10^{-7.3}$ yr^{-1}. In a more recent effort, the same system was integrated for 1 Byr using a symplectic mapping method (Wisdom & Holman 1991). In this work, Pluto's Lyapunov exponent was reported to be "consistent with" that obtained in Sussman & Wisdom (1988). Most recently, several different numerical experiments were reported ill Sussman & Wisdom] 992. In on c of these, all 9 planets were integrated for 1 00 Myr; in the other experiments, the four outer planets only, together with a massless 1 'lute, were integrated for time periods rangi ng from 250 Myr to 1 1 Byr. Each of these runs yielded a positive Lyapunov exponent for Pluto, with Lyapunov timescale between 10 Myr and 20 Myr. The Value of tile Lyapunov exponent obtained is evidently sensitive to the integration met hod and step size used, as well as to slight differences in the modeling and in planetary ma sses and initial conditions.

The chaotic character of a dynamical system also manifests itself in the power spectrum of its dynamical variables. For regular (quasiperiodic) motion, the power spectrum has discrete lines composed of linear combinations of the fundamental frequencies of the system. However, for irregular (chaotic) motion, the]) ower spectrum has a broad band component. in their 845 Myr integration, Sussman & Wisdom (1988) reported just the latter type of spectrum for Pluto's $h = e \sin \varpi$, thus providing corroboration for the chaotic character of their numerical solution for Pluto's orbit.

Stable chaos

The determination of a positive Lyapunov exponent for a dynamical system is usually a quantitative confirmation of chaos that is readily apparent in the time evolution of its dynamical variables. However, Pluto's orbit has now been integrated for 30 100 times its Lyapunov time, and yet no obvious chaotic behavior is to be found in the evolution of its orbital elements.

Could Pluto's motion be a case of "stable chaos"? An example of "stable chaos" has been reported recently in a numerical study of the long term evolution of asteroid 522 Helga (Milani & Nobili 1992). This asteroid lids a Lyapunov time of only 6900" years, yet its orbit remains narrowly confined for more than 1000 times its Lyapunov timescale. Other examples are described in Gladman (1993) where chaotic orbits are found to be bounded for times as long as 105 Lyapunov times! Perhaps this should not come as a complete surprise: a positive Lyapunov exponent is a measure of a local inst ability only;

it does not necessarily imply large scale chaotic behavior.

All the long integrations to date show no large scale instability for Pluto's motion on billion-year timescales. In their 845 Myr integration, Sussman & Wisdom (1988) reported that the divergence of two initially nearby Plutes saturates at a distance of 45 AU. It was pointed out by Milani *et al.* (1989) that this saturation should be expected if the different Plutes remain in approximately the same orbits, but simply diverge in phase while preserving the libration of ϕ with an amplitude near 80°. This suggests that the chaos detected by Sussman & Wisdom does not affect the stability of the 3:2 resonance libration. Milani *et al.* have argued that the origin of the chaos is one of the weaker super-resonances and that the positive Lyapunov exponent indicates that Pluto may be near a chaotic orbit associated with that resonance. Therefore, if Pluto does indeed live in a chaotic zone, that zone is exceedingly narrow. Whether it is connected to a larger chaotic zone (which would allow large scale chaotic changes in its orbit, such as happens to asteroids near the 3:1 Kirkwood Gap (Wisdom 1988)) remains an open question. A final point to note here is that the Sussman & Wisdom calculations used a value for Neptune's mass which is off by about 0.5% from the corrected post-Voyager value; we cannot predict how this correction would affect the Lyapunov exponent calculation.

5. Origin of Pluto's orbit

According; to the accepted paradigm for the origin of the Solar System, the planets accumulated in a flattened disk of dust and gas orbiting the young Sun approximately 4-5 Byr ago. Internal dissipative processes efficiently damped the random non-circular and out-of-plane motions of the forming planets, and, as a result, the major planets move on nearly circular and co-planar orbits. Mercury and Pluto are the striking exceptions to this general rule, Pluto being the more extreme case.

The earliest speculation about the origin of Pluto was a suggestion by Lyttleton (1936) that Pluto may have been a satellite of Neptune which escaped into a heliocentric orbit due to a rare catastrophic event. The main observations that led to this suggestion were that Pluto's orbit crosses that of Neptune, and that Neptune itself possesses a large satellite, Triton, similar in brightness to Pluto. Various means of accomplishing such escape have been considered in the literature (Lyttleton 1936, Horedt 1974, Harrington & van Flandern 1979, Farinella *et al.* 1979, Dormand & Woolfson 1980). This hypothesis has fallen out of favor in recent years as a result of improved knowledge about the characteristics of the Pluto-Charon system which strongly support its formation in a heliocentric orbit. The detailed arguments are reviewed in the chapter by Stern, McKinnon & Lunine.

If Pluto's formation was similar to that of the other planets, it would have formed in a near-circular, low inclination) orbit about the Sun. Indeed, it may have been one of several small icy planets that formed in the outer planetary region. Its peculiar orbit must then be explained as a result of post-formation dynamical processes. Here we summarize two scenarios proposed recently which appear promising. In both these scenarios, Pluto formed in a low-e, low-i orbit beyond Neptune, and outside the 3:2 resonance; and both require a dissipative process o evolve Pluto into its resonant Neptune-crossing orbit.

A. *Chaos -ı Giant impact*

A scenario that has been around in a general way is the Darwinian s~II~~iI'al-c)f-t llc-fittest: namely, that Pluto was one of a swarm of similar small bodies which were continuously scattered by their mutual collisions into and out of the 3:2 resonance with Neptune; Pluto simply happened to be the one that survival to the present time in its protected orbit, while the other bodies were removed by collisions with the giant planets. The existence of Triton, Pluto and Charon lends support to the idea that there were other similar ice-dwarf planets in the outer Solar System.

Numerical calculations by Applegate *et* 01. (19S6), Kinoshita & Nakai (I 9S4), and Olson-Steel (1988) have indicated that, given the current masses and orbits of the giant planets, most Plutc-like orbits near the 3:2 Neptune resonance exhibit large scale chaotic variations over very snort timescales, $\sim 10^5$yr. Recent numerical integrations by Holman & Wisdom (1993) and Levison & Stern (1993) have further explored the orbital dynamics near the 3:2 Neptune resonance. These authors find that test particles placed in initially low-eccentricity, low-inclination orbits near this resonance evolve rapidly (on $\sim 10^6$yr timescales) into orbits with high eccentricity and inclination similar to that of Pluto. Furthermore, Levison & Stern find that some of these orbits exhibit resonant librations of ϕ, but with large amplitudes that vary chaotically between 100° and 120°. The dynamical lifetime of such orbits (before a close approach to Neptune) was found to be short, $\mathcal{O}(10^7)$ yr, although in some cases, the time to first Neptune encounter was almost 1 Byr.

Levison & Stern suggest that a "Pluto" on such a chaotic orbit may be stabilized by means of a collision with a smaller mass body. In their scenario, the collision results in the formation of the Pluto-Charon binary; the orbit stabilization is accomplished if the collision knocks the libration amplitude of ϕ to a value smaller than $\sim 80°$.

This proposal is attractive for it accounts for the high eccentricity and inclination of Pluto's resonant orbit. However several questions regarding its plausibility remain: (*i*) Pluto's initial low-e, low-i orbit is required to be in a very narrow range of a; (*ii*) the underlying dynamical mechanism for the excitation of e and i is not known; the numerical calculations rely upon the current configuration of the giant planets; is the e, i excitation mechanism sufficiently insensitive to the evolution of the outer planet masses and orbits during their formation? (*iii*) the window-of-opportunity for a Charon-forming impact is a short period of time after Pluto becomes Neptune-crossing and before it has a close encounter with Neptune; (*iv*) the type of collision that would stabilize the chaotic orbit (a gentle "nudge" that places the planet close to the edge but inside the stable libration zone of the 3:2 Neptune resonance) is different in character from the giant impact required for the formation of the Pluto-Charon binary; the former indicates a very small impactor-to-target mass ratio, whereas the giant-impact origin for the Pluto-Charon binary involves nearly ex-equal colliding bodies. Finally, the origin of the ω libration and the other secular resonances of Pluto remain undetermined in this scenario. Further studies on this origin model will doubtless prove valuable.

B. Resonance capture

Phase-locking as a result of some slow dissipative process is a phenomenon well-known in nature. in the Solar System, orbit-orbit resonances (as well as spin-orbit resonances) are commonly found amongst the satellites of the giant planets. Capture into an orbit-orbit resonance occurs when the orbits of two bodies approach each other through some dissipative process. The origin of resonances amongst planetary satellites is thought to be due to tidal friction and is a well understood phenomenon (Peale 1986). One of us (Malhotra (1993)) has proposed that Pluto may have been captured into the 3:2 Neptune resonance during the late stages of planet formation, when Neptune's orbit expanded as a result of angular momentum exchange with residual planetesimals.

The mechanism is summarized as follows. The giant planets' gravitational effects cleared out their inter-planetary regions by scattering the unaccreted mass of planetesimals. (Some fraction of this mass now resides in the Oort Cloud of comets in a roughly isotropic distribution surrounding the planetary system (Weissman 1990), but most has been lost from the planetary system.) A planetesimal scattered outward gains angular momentum, while one scattered inward loses angular momentum at the expense of the planets. The back reaction of planetesimal scattering on the planets caused the planet orbits to evolve.. Consider the effect on Neptune: approximately equal numbers of planetesimals are scattered inward as outward; those planetesimals scattered outwards by this planet end up in the Oort Cloud, or return to be re-scatted. A fraction of the latter set is again scattered inwards. The inwardly scattered planetesimals are systematically ejected from the planetary system by the more massive Jupiter which resides interior to Neptune. in effect, the outer less massive planets, Neptune and Uranus, transfer control of the ejection process to the inner more massive Jupiter (and to a lesser extent, Saturn). Numerical simulations confirm this general picture: Neptune, Uranus and Saturn's orbits expand, while Jupiter's orbit shrinks (Fernandez & Ip 1984).

If Pluto were initially in a nearly circular and co-planar orbit beyond the orbit of Neptune, then as Neptune's orbit expanded, its exterior orbital resonances approached Pluto. In particular, if Pluto's initial orbital radius were such that the 3:2 resonance was the first major Neptune resonance to sweep by, Pluto would be captured into this resonance. The resonance capture occurs with 100% probability if Pluto's initial eccentricity were smaller than ~ 0.03; the capture probability is smaller for higher initial eccentricities (10% for $e_{\text{initial}} \approx 0.15$). In the subsequent evolution, as Neptune's orbit continued to expand, the resonant perturbations increased Pluto's orbital eccentricity. Malhotra finds the following relation between Pluto's eccentricity and Neptune's semimajor axis:

$$e^2_{\text{final}} - e^2_{\text{initial}} \approx \frac{1}{3} \ln \left(\frac{a_{\text{N,final}}}{a_{\text{N,initial}}} \right). \tag{5}$$

This equation shows that Pluto's current eccentricity would have been produced by its capture into the 3:2 resonance when Neptune's semimajor axis was approximately 25 AU (~ 5 AU less than its current value). By implication, Pluto's initial orbital radius was ~ 33 AU. This equation also indicates a rather weak dependence of the final eccentricity on the initial a and e. We note that in this scenario, Pluto is initially not in a Neptune-crossing orbit; although the resonance forces the high eccentricity on Pluto, it also provides protection against close approaches during the entire evolution.

Malhotra's numerical simulations confirm the feasibility of this theory for accounting for the observed eccentricity as well as the observed libration amplitude of ϕ. But the theory is not complete: Pluto's high inclination, the ω libration, as well as its other secular resonances remain to be explained. The magnitude of the radial migration of Neptune (as well as the other giant planets) implied in this theory has other theoretical and observational consequences that remain to be evaluated. The apparently high efficiency of resonance capture and its implications for the evolution of other small bodies in that region also needs to be examined. The formation of the Pluto-Charon binary pair is not addressed, but that is not a difficulty with the theory; a Pluto-Charon binary formed in a low-e, low-i orbit would undergo the same resonance capture evolution.

6. Summary

The heliocentric orbit of the Pluto- Charon pair has been observed for nearly one-third of its orbit period. Ephemerides of the pair (collectively called "Pluto" in this chapter) fit the known observations wall. However, one must bear in mind that the span of observation is still much less than an orbit period, and there is a need for future positional observations and orbit fits. A spacecraft mission to Pluto would need ephemerides of high accuracy.

Pluto's orbit is the most eccentric and inclined of the major planets (Figure 1). At perihelion it ventures closer to the sun than Neptune, seemingly violating the well-s])aced, hierarchical pattern of the other planets that is associated with long-term orbital stability. Investigation of the dynamical evolution of this configuration shows a surprisingly complex behavior involving at least three resonances.

i. Every 494 yr Pluto orbits the Sun twice while Neptune orbits three times. The resonant perturbations from Neptune cause Pluto's orbital period to librate about 247 yr and the resonance argument (cf. Eq. 1) to librate about 180° with an amplitude of 82°. The libration period is 19,912 yr. This resonance prevents Pluto from passing close to Neptune; when the two planets have the same heliocentric longitude Pluto is closer to its aphelion than its perihelion (Figure 2). So effective is this resonance at keeping the two planets apart that Pluto can approach Uranus more Closely than Neptune. The 3:2 resonance is a strong stabilizing influence on orbit. Indeed, integrations of Pluk-like orbits just outside of the resonance libration region display strong chaotic behavior in only a few million years.

ii. Pluto's argument of perihelion dots not precess through 360°; rather, it librates about a value of 90°. The libration period is 3.8 Myr and its amplitude averages 21°. Thus, Pluto's perihelion and aphelion never cross Neptune's orbit plane. This also helps keep the two planets apart.

iii. The difference between Pluto and Neptune's nodes circulates every 3.8 My r the same period as the argument of perihelion libration. This is another resonance and it has an associated libration period of 34 Myr.

Pluto is near a fourth resonance, but the differences between independent integrations can support either an exact resonance with a librating argument, or a near-resonance with a circulating argument. The difference of the longitudes of perihelion of Neptune and Pluto circulates in 1.267 Myr - one-third of the period of the argument of perihelion libration, and also of the circulation period of the node difference. Because resonance separatrices are usually associated with chaotic zones, there will be uncertainty about this potential source of chaotic behavior until Pluto's true placement with respect to this resonance is known.

That Pluto's orbit is chaotic is indicated by the positive Lyapunov exponent determination in several different long numerical integrations. However, the integrations which indicate the presence of chaos do not show obvious erratic changes in the orbital elements that one might expect with the short Lyapunov time of $\mathcal{O}(10^7)$yr. Indeed, all the evidence suggests that the protection accorded Pluto by the 3:2 Neptune resonance is robust over billion-year timescales. There remains a degree of uncertainty in the magnitude of the Lyapunov timescale - slightly different values obtained in different numerical integrations. The origin of the chaos remains unknown, as well. The near-resonance described above is one possible but unproven source of the chaos; it does illustrate an important point: the magnitude of the chaotic orbital perturbations may be small and bounded, or, if unbounded, may require times much longer than the age of the solar system to be dangerous to Pluto's existence. Nevertheless, a Lyapunov timescale of $\mathcal{O}(10^7)$yr implies a relatively short horizon of predictability for its exact position and velocity.

Is it significant that small differences in the modeling or integration methods result in Lyapunov exponents differing by a factor of several? If Pluto's orbit is chaotic, what is the origin of the chaos, and how is it manifested in its orbital element evolution? Because Pluto's chaotic motion may be associated with a narrow chaotic zone, it is desirable to eliminate the uncertainties in past numerical integrations due to (now known) errors in planetary masses and orbital initial conditions. The Voyager flybys of the outer planets have provided a much improved set of outer planet masses and continued analyses of positional data sets has provided compatible ephemerides for the planetary initial conditions. Future numerical integrations should take advantage of these improvements.

What role do the various resonances play in the origin and continued existence of the Pluto-Charon system? Because some of the resonances are protective in that they increase the minimum distance between Pluto and Neptune, has a dynamical "survival of the fittest" left objects only in protected orbits? Or, might the resonances have captured these (and perhaps also other) objects that did not start out in resonant orbits? We have discussed two origin scenarios proposed recently, one in each of these two categories. Both these scenarios suggest the formation of Pluto in an initially non-resonant, nearly circular and co-planar orbit in the outer reaches of the Solar System, beyond the orbit of Neptune. One invokes a catastrophic impact to emplace Pluto in its stable resonant orbit; the other suggests a slow evolution of the orbit during the late stages of formation of the outer planets. Much work remains to be done to establish the details of each of these models. Detailed studies of the dynamics of each of Pluto's resonances will help to evaluate the relative merits of these models. They can also be expected to provide clues to the underlying causes of the large-Lyapunov-exponent-without-large-scale-chaotic-behavior. In turn, the orbit of this pair of bodies at the edge of the planetary system may hold clues to the early dynamical evolution of the Solar System.

The fascinating dynamical complexity of Pluto's orbital evolution could not have been guessed at its first sighting as a moving point of light in 1930. A large number of questions invites further study of the Pluto-Charon pair.

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FIGURE CAPTIONS

Figure 1. *The orbits of the outer planets in a heliocentric reference frame.* (a) Projection in the plane of the ecliptic. (b) Projection in cylindrical polar coordinates (r is the distance from the Sun projected in the ecliptic, and z is the distance above the ecliptic).

Figure 2. *The orbits of the outer planets for 40,000 years in a reference frame co-rotating with the mean motion of Neptune.* In this reference frame two orbits of Pluto trace out a complete loop in $\sim$ 500 yr. This figure visualizes the effects of the most important resonance of Pluto's motion: the 3:2 resonance libration which causes Pluto's heliocentric longitude conjunctions with Neptune to librate about aphelion with a period of about 20,000 yr; its perihelion librates 90° away from Neptune with that same period.

Figure 3, *The distance between Neptune and Pluto.* The synodic period of Neptune and Pluto is $\sim$ 500 yr. Their closest approach distance (near aphelion) varies between $\sim$ 17 AU and $\sim$ 22 AU.

Figure 4. *The distance between Uranus and Pluto.* The synodic period of Uranus and Pluto is $\sim$ 130 yr. The closest approach distance between these planets varies between $\sim$ 12 AU and $\sim$ 26 AU.

Figure 5. *The resonance argument,* $\phi = 3\lambda - 2\lambda_N - \omega$ (in degrees) as a function of time, in years, from the present. (Reproduced, with permission, from Milani *et al.* 1989).

Figure 6. *Pluto's argument of perihelion,* ω for 2?14 My'. The abscissa is time, in days. The 3.8 Myr libration is modulated with a 34 Myr period. (Reproduced, with permission, from Applegate *et al.* 1986).

Figure 7. *The finite-time Lyapunov exponent* of Pluto, γ as a function of time. In this log-log plot, convergence to a positive Lyapunov exponent (for a chaotic trajectory) is indicated by a leveling off"; for a regular trajectory, this trace would approach a straight line with slope - 1. (Reproduced, with permission, from Sussman & Wisdom 1988).

8. *Surface of section* in the circular planar restricted 3-body model for the Neptune-Pluto 3:2 resonance. The parameters of the model are $m_N/m_\odot = 5.146 \times 10^{-5}$, the Sun-Neptune separation is 1.0, and the Jacobi integral has the value $C \simeq -1.4899$; the dynamical variables are $\phi = 3\lambda - \lambda_N - \varpi$, $J = \sqrt{a}(1 - \sqrt{(1 - e^2)})$, where a is in units of the Sun-Neptune distance.

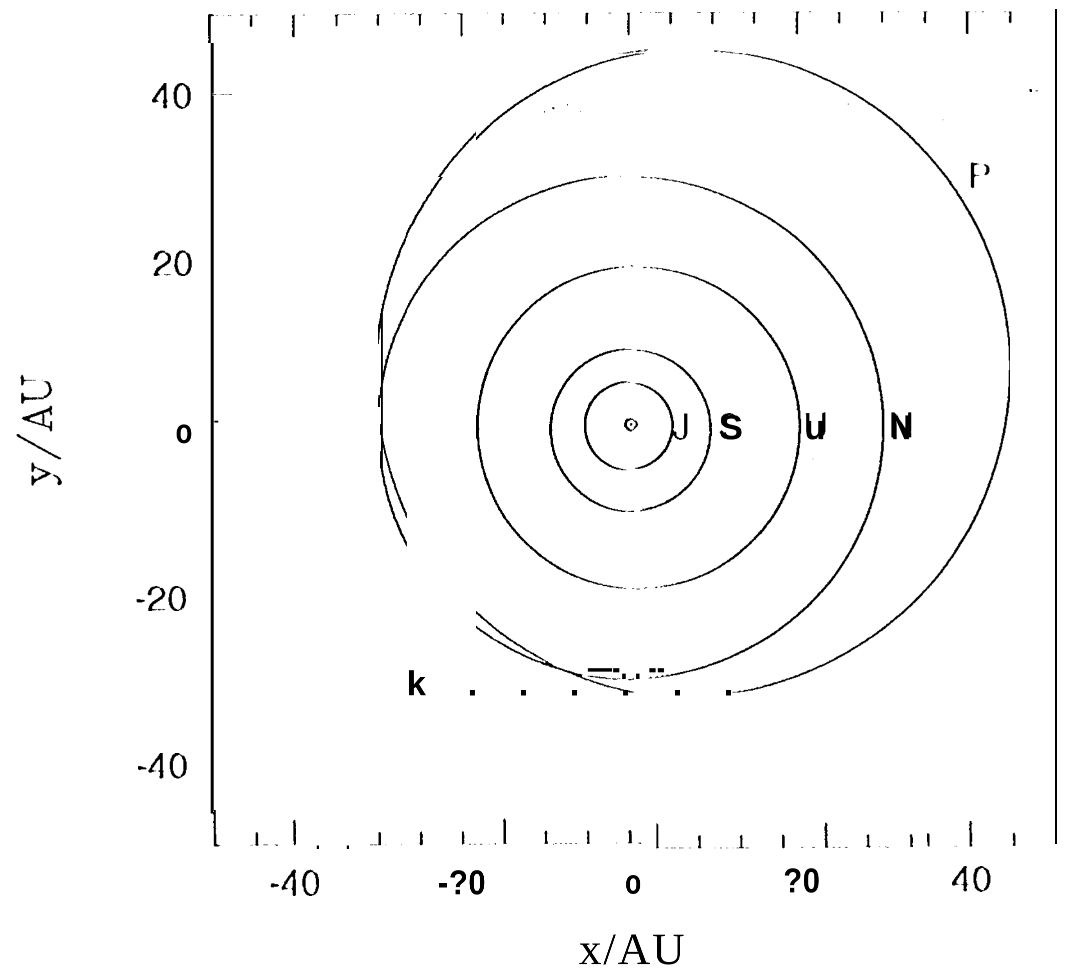


Fig. 1(a)

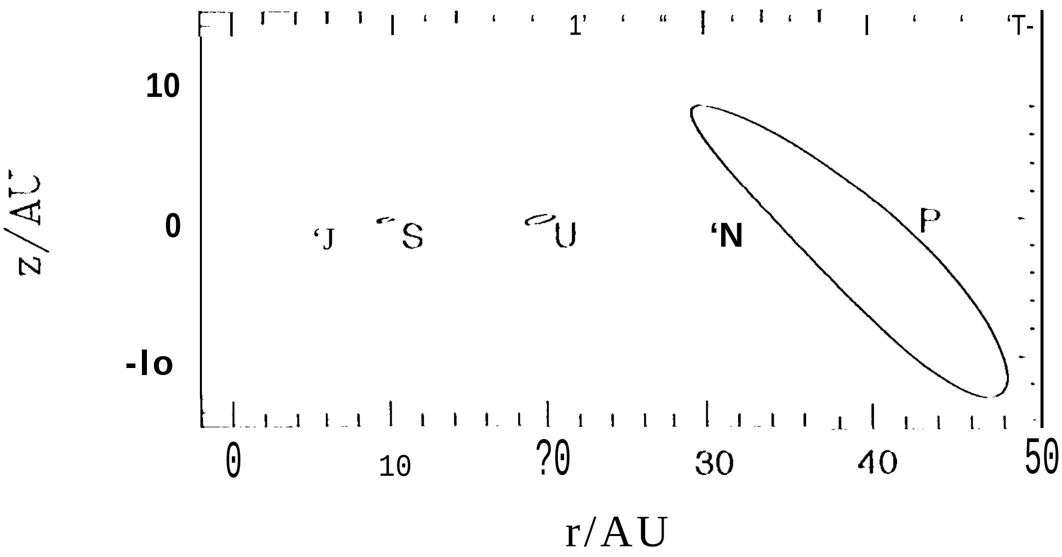


Fig. 1(b)

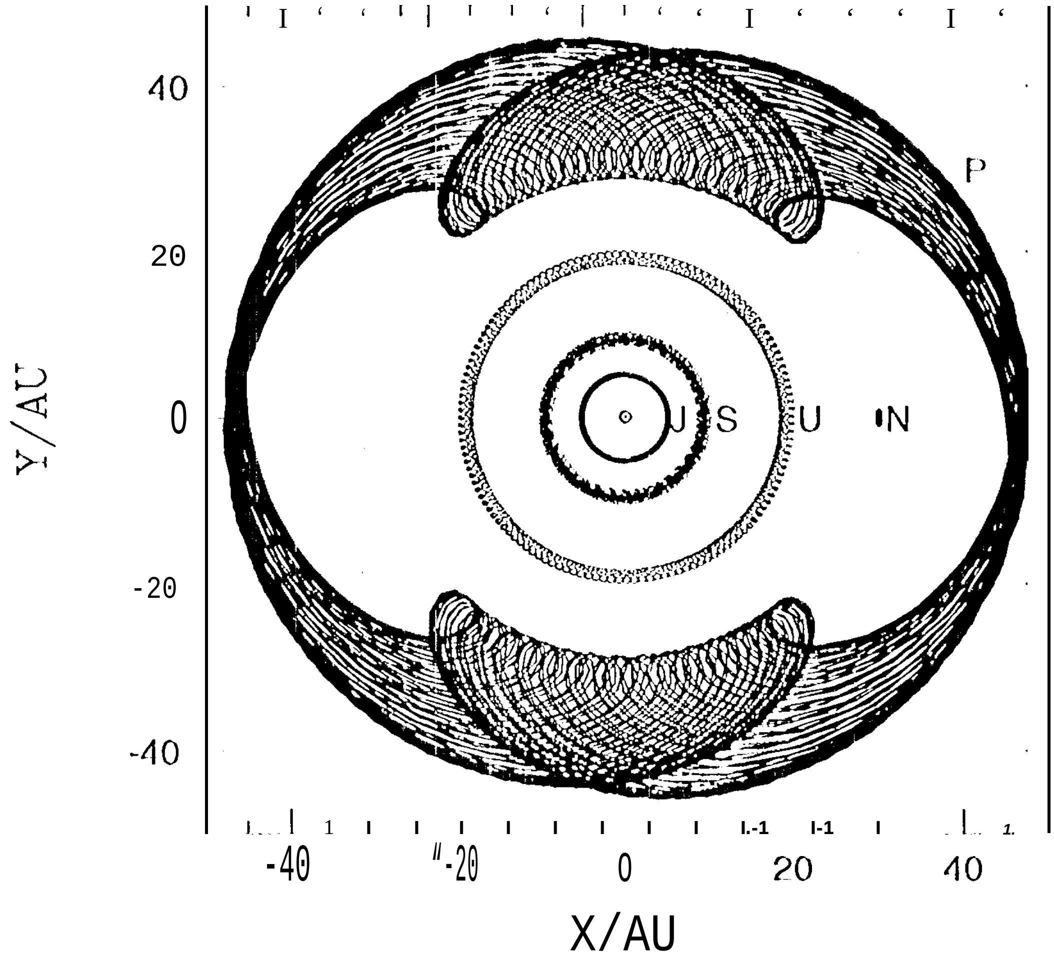


Fig. 2

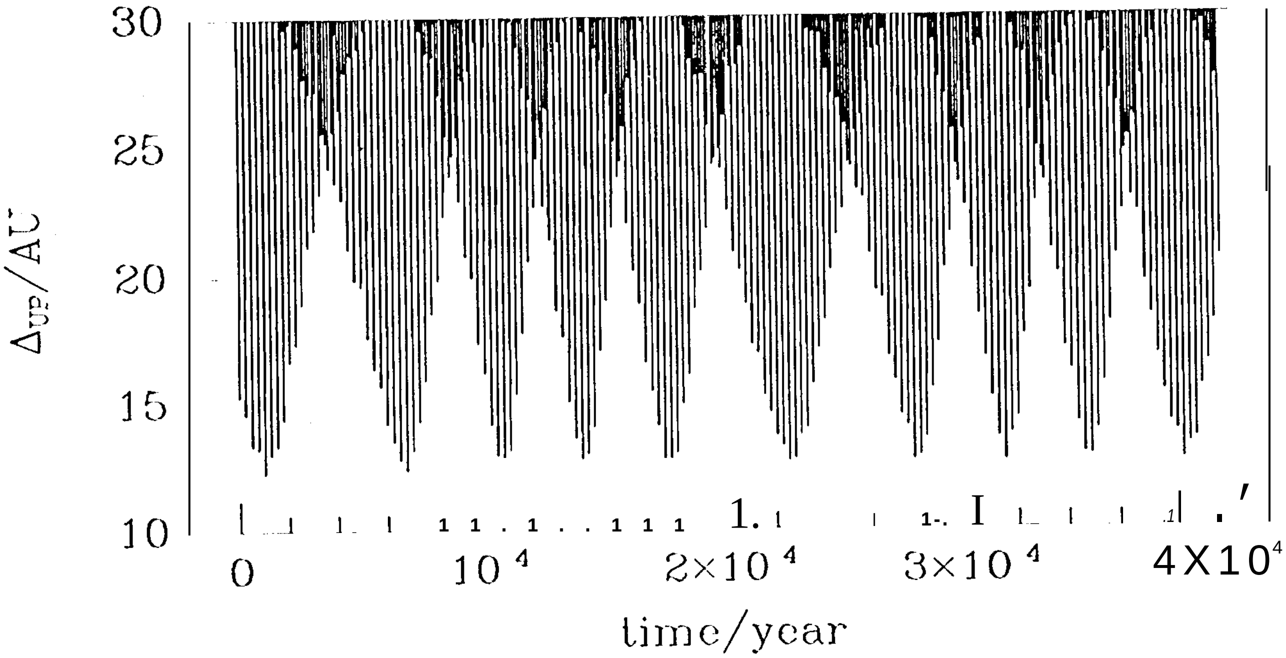
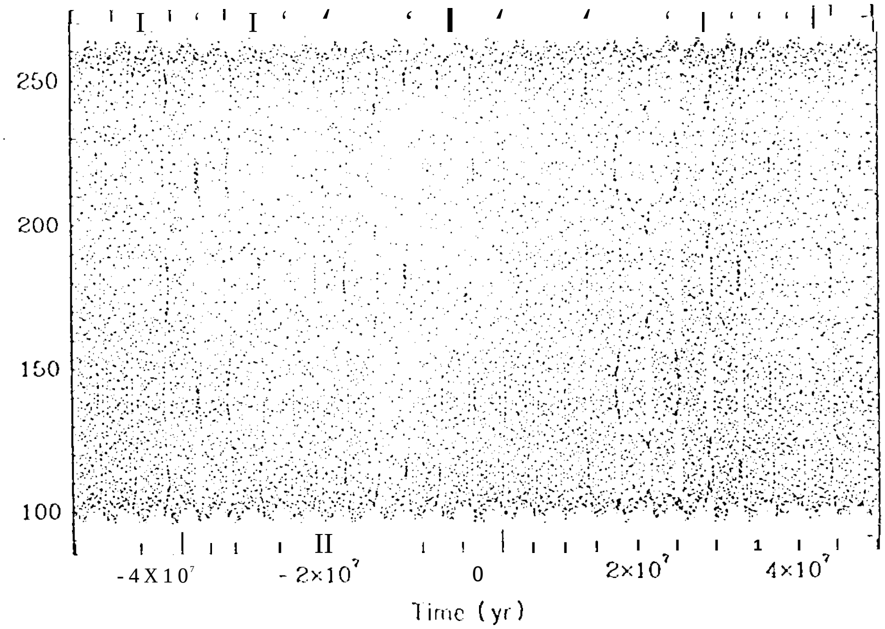
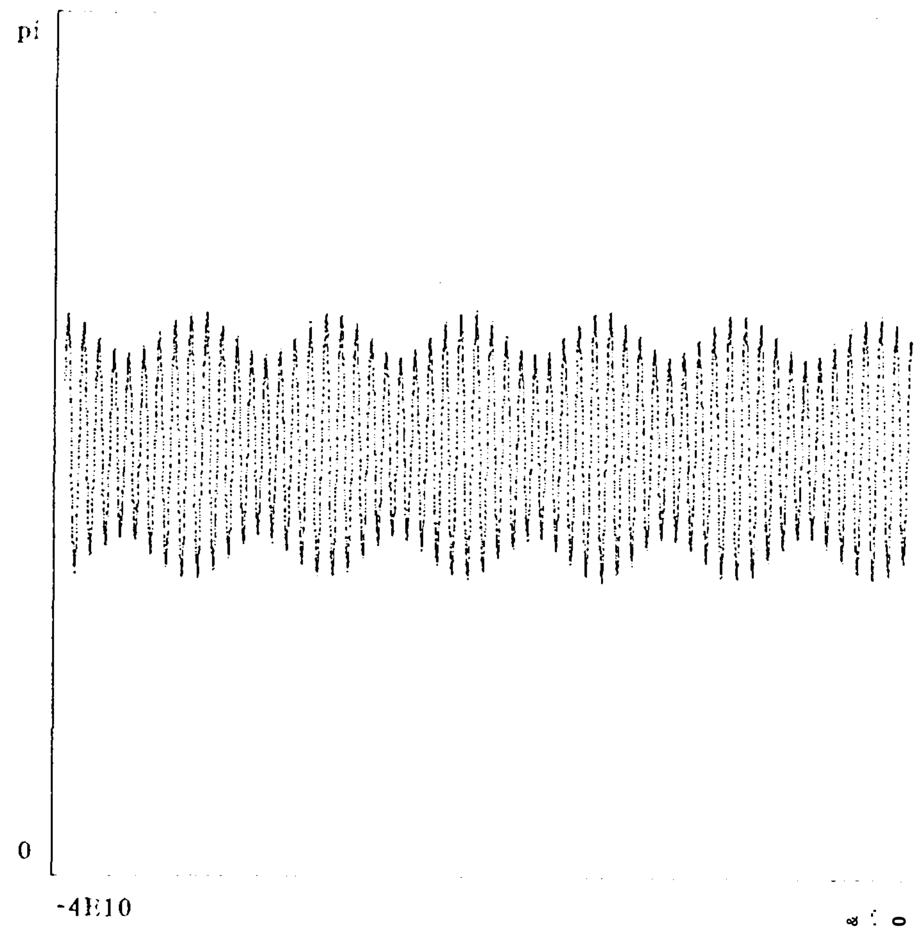

30
25
20
15
10
Δ_{UP}/AU
0
10^4
2×10^4
3×10^4
4×10^4
time/year

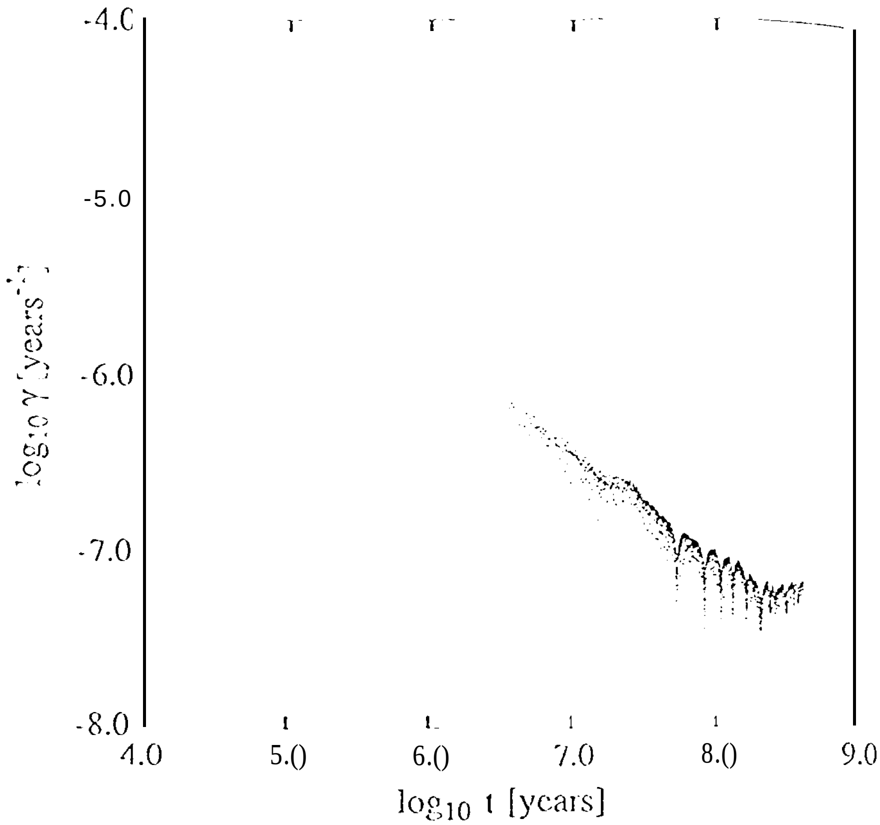
Fig. 4



250
200
150
100
I
I
II
-4X10^7
-2×10^7
0
2×10^7
4×10^7
Time (yr)



pi
0
-4E10
0



-4.0
-5.0
-6.0
-7.0
-8.0
4.0
5.0
6.0
7.0
8.0
9.0
log10 t [years]

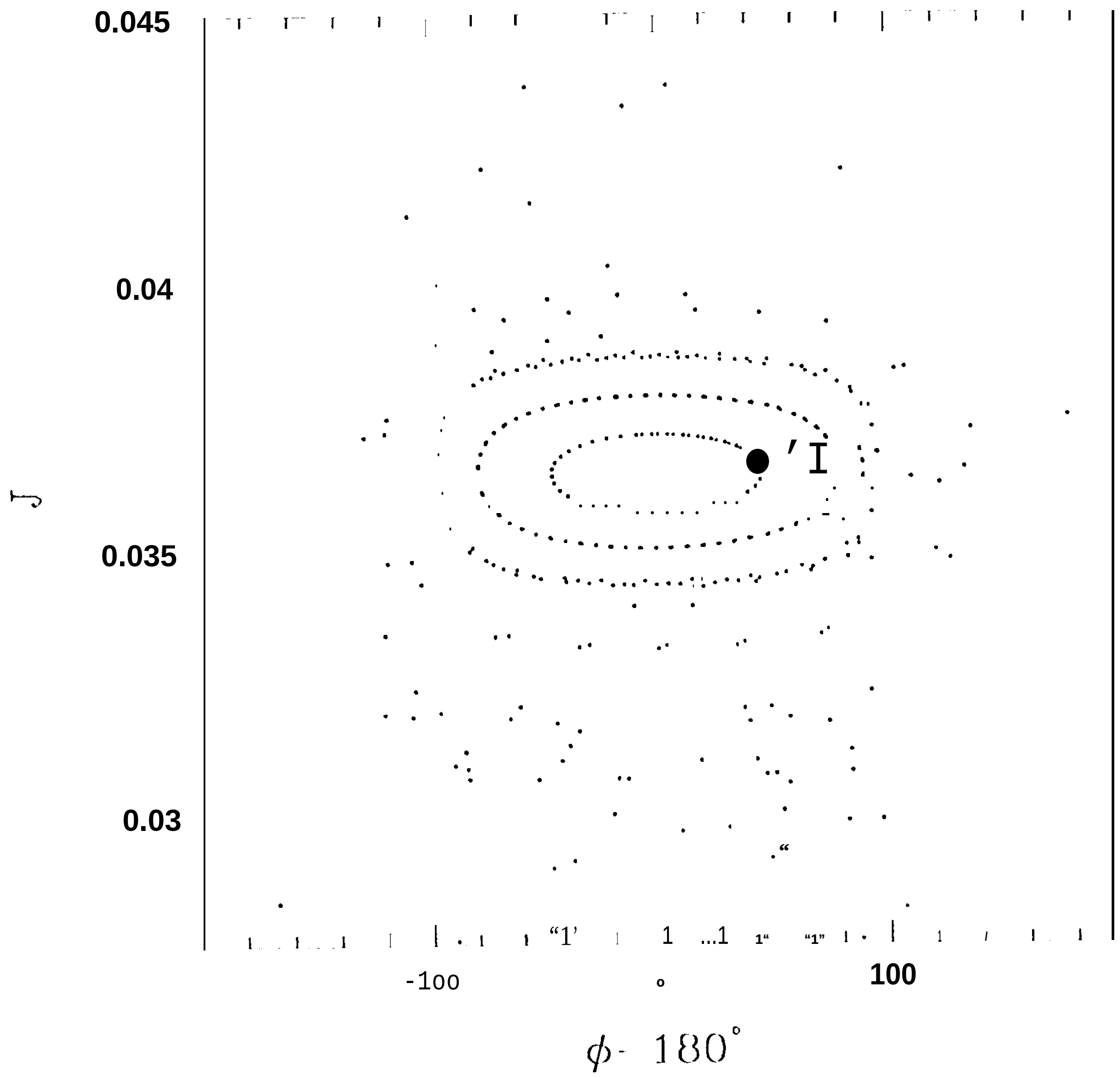


Fig. 8

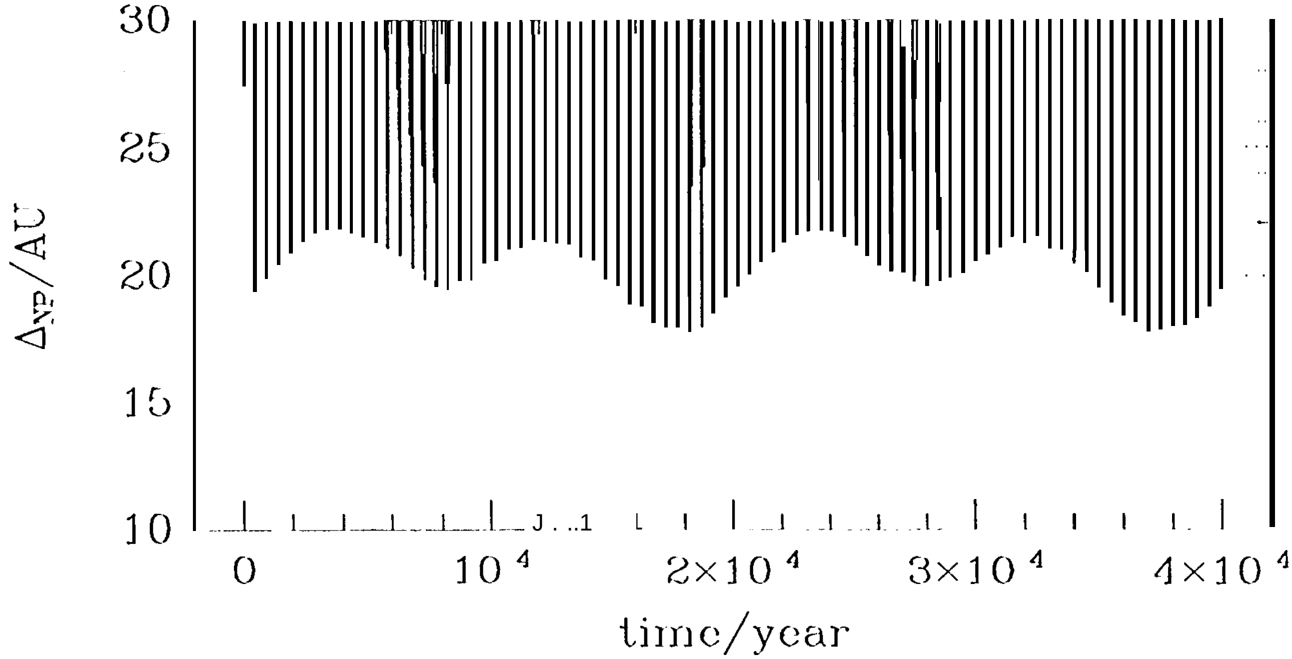

30
25
20
15
10
Δ_{VP}/AU
0
10^4
2×10^4
3×10^4
4×10^4
time/year

Fig. 3