Wavefront control algorithms and analysis for a dense adaptive optics system

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ABSTRACT

This paper presents the development and analysis of a wavefront control strategy for the Space Laser Electric Energy (SELENE) power beaming system. SELENE represents a substantial departure from most conventional adaptive optics systems in that the deformable element is the segmented primary mirror and the signal that is fedback includes both the local wavefront tilt and the relative edge mismatch between adjacent segments. The major challenge in designing the wavefront control system is the large number of subapertures that must be commanded. A fast and near optimal algorithm based on the local slope and edge measurements is defined for this system.

1. SUMMARY

This report presents the development and analysis of a wavefront control scheme for SELENE. The controller is derived from the assumption that the wavefront is locally flat over each subaperture. This assumption leads to a strategy that involves a two step implementation requiring first the local correction for wavefront tilt for each subaperture, followed by a global correction for the piston error. Conventional AO systems employing a continuous deformable mirror as the correcting optical element achieves the piston correction via a wavefront reconstruction process based on local gradient (tilt) information. The reconstruction process typically leads to a discretized Poisson equation with normal boundary conditions to estimate the wavefront. Because the adaptive optical element for SELENE is not a continuous surface, a slightly different path must be taken to reconstruct the wavefront. The reconstruction process for SELENE entails the use of edge displacement measurements to supplement the tilt measurements to fill in the gaps, so to speak, created by the discontinuous surface. SELENE wavefront reconstruction is shown to lead to a discretized Poisson equation as well. The derived control law turns out to be identical to the control strategy that has been pursued since the inception of the SELENE program: (i) correct for wavefront tilt., (2) minimize the edge mismatch error in a least squares sense[1]. Thus our analysis essentially provides a new interpretation ofthis control strategy,

This new interpretation facilitates several developments. Firstly, we show the optimality of the algorithm (i.e., the conditions under which it is optimal). From there we are able to establish some error bounds for the control algorithm. This in turn allows us to set requirements on the edge sensor. Basically what is shown here is that the rms piston error due to edge sensing is approximately of unity magnitude. We also show that the piston error grows logarithmically with the number of subapertures. Hence, reducing segment size places tighter requirements on the edge sensor, although rather mildly. The effect of reducing segment size actually has a more substantial effect on the reconstruction error due to tilt error. This growth turns out to be linear with decreasing subaperture size.

An efficient implementation of the reconstruction algorithm is developed that requires $O(N^2 log N)$ floating point operations to implement, where N^2 denotes the number of subapertures. The algorithm is based on embedding (regularizing) the Poisson problem into a problem defined on a square. Fast solvers (i.e., those requiring $O(N^2 log N)$) flops exist for solving this problem2 Bounds on the increased covariance of the wavefront estimate introduced by the embedding procedure are presented. These bounds indicate the increased error to be rather benign, perhaps a .13 increase for systems the size of SELENE. (The larger the system, the smaller the error.)

2. AN IDEALIZED PROBLEM

Let $w(x)$ denote the instantaneous wavefront, and let the segmented primary surface be represented by the piecewise linear function $u(x)$,

$$u(x) = \sum_i \chi(\Lambda_i) u_i(x), \tag{1}$$

where Λ_i denotes the i^{th} segment, $\chi(\cdot)$ = characteristic function ($\chi(\Lambda_i)(x) = 1$ if $x \in \Lambda_i$, zero otherwise), and $u_i(x)$ is linear. Let x_i be the centroid of Λ_i. Ideally we would like to minimize the wavefront error J,

$$J = \int_\Lambda |w(x) - u(x)|^2 dx, \quad \Lambda = \cup \Lambda_i. \tag{2}$$

From (2) we write

$$J = \sum_i \int_{\Lambda_i} |w(x) - u_i(x)|^2 dx, \tag{3}$$

and note that it is sufficient to independently minimize the error for each segment.

Now given that u_i is linear, and assuming that w is C^2 (two continuous derivatives), a reasonable objective for the control scheme is to choose u_i so that

$$u_i(x) = w(x_i), \quad \text{and} \quad \nabla u_i(x_i) = \nabla w(x_i). \tag{4}$$

By satisfying (4) we have for $x \in \Lambda_i$,

$$|w(x) - u_i(x)| \le 1/2 \max_{\xi \in \Lambda_i} < (x - x_i), W(\xi)(x - x_i) >, \tag{5}$$

where $W(\xi)$ denotes the Hessian of w. hTow if we assume that the distance between adjacent centroids is h, and the area of the total aperture is d^2, the number of segments comprising the primary, call it N is of order $N = O(d^2/h^2)$. The error J in (2) is now approximated as

$$\begin{aligned} J &= \sum_i \int_{\Lambda_i} |w(x) - u_i(x)|^2 dx \\ &\le 1/2 \sum_i \max_{\xi \in \Lambda_i} |W(\xi)|^2 \int_{\Lambda_i} |x - x_i|^4 dx \\ &\approx d^2 O(h^4), \end{aligned} \tag{6}$$

since

$$\int_{\Lambda_i} |x - x_i|^4 dx = O(h^6)$$

and $N = O(d^2/h^2)$. Hence, the normalized rms wavefront error is

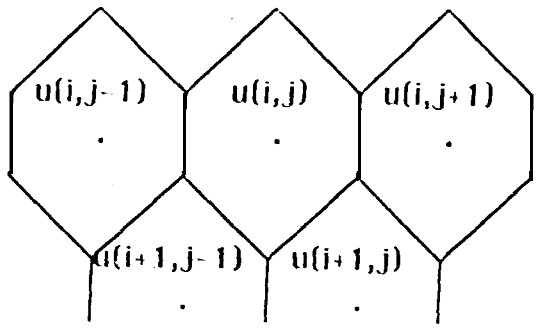
$$\sqrt{J}/d^2 = O(h^2) \tag{7}$$

where the constant is of the order $sup_{\xi \in \Lambda} W(\xi)$. We note that this controller essentially corrects for the piston, tip, and tilt across each subaperture. Noll[3] has dervied an expression for the error as a function of the residual uncorrected Zernike terms of the disturbance, and in this case has shown the residual error to be approximately .13 radians^2 of phase based on a Kolmogorov turbulence spectrum.

3. THE NONIDEALIZED PROBLEM

The idealized situation above is characterized by perfect reconstruction of the wavefront $w(x)$ followed by the implementation of the control law defined in (4). This controller presupposes both global knowledge of the wavefront *and* of the aperture function $u(x)$. In the true SELENE configuration neither of these is available. But before before considering this configuration, we will first treat an intermediate case between the ideal and actual to show how edge sensing contributes to the wavefront reconstruction problem.

The SELENE hexagonal segments are arranged as in the figure below:



We take the distance between adjacent centroids to be h. Now we make the following assumptions:

(i) $w(x)$ must be estimated from the wavefront tilt measurements $\nabla w_{ij} = \nabla w(x_{ij})$.

(ii) u_{ij} can only be estimated from edge displacement measurements, ϵ_{ij}, and segment tilt measurements $\gamma_{ij} = \nabla u(x_{ij})$. Here ϵ_{ij} and γ_{ij} are both 2 - *vectors*, $\epsilon_{ij} = [\epsilon^x_{ij} \quad \epsilon^y_{ij}]$, and $\gamma_{ij} = [\gamma^x_{ij} \quad \gamma^y_{ij}]$. (In the SELENE setting we only measure the difference $\nabla w_{ij} - \gamma_{ij}$. This case will be taken up shortly.)

Let $\hat{w}$ denote a least squares or minimum variance estimate of w. (Without loss of generality we will assume that $\hat{w}$ has been normalized so that $\sum_{ij} \hat{w}_{ij} = 0$.) The geometry in the figure leads to the edge displacement relationship

$$\epsilon^x_{ij} = u_{ij+1} - u_{ij} - h/2(\gamma^x_{ij+1} + \gamma^x_{ij}) \tag{8}$$

for horizontally adjacent segments; and for diagonally adjacent segments

$$\epsilon^y_{ij} = u_{i+1\,j} - u_{ij} + \frac{h}{2\sqrt{2}}(\gamma^y_{i+1\,j} + \gamma^y_{ij} - (\gamma^x_{i+1\,j} + \gamma^x_{ij})). \tag{9a}$$

Because each segment of SELENE has edge sensors on every side, there is another measurement corresponding to the southwest diagonal,

$$\epsilon^s_{ij} = u_{i+1\,j-1} - u_{ij} + \frac{h}{2\sqrt{2}}(\gamma^y_{i+1\,j-1} + \gamma^y_{ij} - (\gamma^x_{i+1\,j-1} + \gamma^x_{ij})). \tag{9b}$$

(We will ignore this measurement in the analysis and algorithm development that follows, but revisit it later.) Introduce the difference operator A,

$$A = \begin{bmatrix} A^x \\ A^y \end{bmatrix}, \tag{10}$$

where

$$A^x u = u_{ij+1} - u_{ij} \tag{10a}$$

and

$$A^y u = u_{i+1\,j} - u_{ij} \tag{10b}$$

for $u = [u_{11} \quad u_{12} \ldots]$, (i.e., stacking u by rows in the array). Then (8)-(9) can be written as

$$Au = b \tag{11}$$

where b is a linear combination of the *measured* tilts. Now let $\hat{u}$ denote a least squares estimate (or minimum variance estimate) of u. We will assume again that $\hat{u}$ has been normalized so that $\sum \hat{u}_{ij} = 0$. In this intermediate case a compensation scheme can be defined by the local tilt command, $\Delta\gamma$,

$$\Delta\gamma = \nabla\hat{u} - \gamma, \quad \gamma \leftarrow \gamma + \Delta\gamma \tag{12}$$

followed by the differential piston command, Δu,

$$\Delta u = \hat{w} - \hat{u}, \quad \hat{u}^+ = \hat{u} + \Delta u. \tag{13}$$

Here τ^+, $\hat{u}^+$ denote the updated tilt and piston vectors. Note again that the differential command Δu requires the global reconstruction of the wavefront w, while $\Delta\tau$ only requires local measurements.

Next we will treat the more general SELENE case where we do not have independent measurements of ∇w and τ, but only their difference y,

$$y_{ij} = \nabla w_{ij} - \tau_{ij} \ . \tag{14}$$

Note that we still have the differential tilt command via (12)

$$\Delta\tau = y, \tag{15}$$

but we cannot use (13) for the differential piston command because the estimates $\hat{u}$, $\hat{w}$ cannot be formed. However, observe that to implement (13) it is only necessary to have an estimate of the *difference* $w - u$. To this end let's assume that (15) has been implemented so that we may write

$$\nabla w = \tau. \tag{16}$$

Now since

$$\tau^x_{ij} = \frac{w_{ij+1} - w_{ij}}{h} + O(h^2), \tag{17}$$

substituting (17) into (8) gives (neglecting the $O(h^2)$ term)

$$\begin{aligned} \epsilon^x_{ij} &= u_{ij+1} - u_{ij} - \frac{h}{2}\left[\frac{w_{ij+2} - w_{ij+1}}{h} + \frac{w_{ij+1} - w_{ij}}{h}\right] \\ &= u_{ij+1} - u_{ij} - 1/2[w_{ij+2} - w_{ij}]. \end{aligned} \tag{18}$$

Also observe that

$$\tau^y_{ij} = \frac{\sqrt{2}}{h}\left[w_{ij} - w_{i+1j} + \frac{h}{\sqrt{2}}\tau^x_{ij}\right] \tag{19}$$

since

$$w_{i+1j} = w_{ij} + \frac{h}{\sqrt{2}}(\tau^x_{ij} - \tau^y_{ij}). \tag{20}$$

Hence,

$$\tau^y_{ij} = \frac{\sqrt{2}}{h}[w_{ij} - w_{i+1j}] + \tau^x_{ij}. \tag{21}$$

And consequently,

$$\begin{aligned} \epsilon^y_{ij} &= u_{ij+1} - u_{ij} + \frac{h}{2\sqrt{2}}\Big[\frac{\sqrt{2}}{h}(w_{i+1j} - w_{i+2j}) + \tau^x_{i+1j} + \frac{\sqrt{2}}{h}(w_{ij} - w_{i+1j}) \\ &\quad + \tau^x_{ij} - \tau^x_{i+1j} - \tau^x_{ij}\Big] \\ &= u_{ij+1} - u_{ij} - 1/2[w_{i+2j} - w_{ij}]. \end{aligned} \tag{22}$$

Now, $w_{ij+1} - w_{ij}$ and $w_{i+1j} - w_{ij}$ are close approximations to $1/2(w_{ij+2} - w_{ij})$ and $1/2(w_{i+2j} - w_{ij})$, respectively. In fact all of these quantities are just difference approximations to either $\partial w/\partial x$ or $\partial w/\partial y$. The magnitude of their difference is consequently $O(h^2)$, with constant again of order $max_{\xi\in\Lambda_i}|W(\xi)|$, i.e. for example

$$|\,1/2(w_{i+2j} - w_{ij}) - (w_{ij+1} - w_{ij})| < h^2 \sup_{\xi} |W(\xi)|. \tag{23}$$

Putting (18), (22), and (23) together we get

$$\epsilon^x_{ij} = u_{ij+1} - u_{ij} - (w_{ij+1} - w_{ij}) + O(h^2) \tag{24}$$

$$\epsilon^y_{ij} = u_{i+1j} - u_{ij} - (w_{i+1j} - w_{ij}) + O(h^2). \qquad (25)$$

Hence, for small $|h|$,

$$\epsilon \approx A(u - w), \qquad (26)$$

and the least squares (or minimum variance) estimate of $u - w$ can be obtained directly from the edge measurements after the local tilt corrections have been made. Thus the compensation scheme becomes:

(i) Implement the differential tilt command via (15)

(ii) Estimate $\hat{v}$, $v = u - w$, from (26) (more on this step in a little bit)

(iiii) Implement the differential piston command via (13)

It is worthwhile to note that this two step control law can also be interpreted as minimizing the least squares error in the adjacent edge mismatch *after* tilt correction has been made. To see this suppose the differential tilt correction has been made, and now the objective is to implement a piston command to minimize the edge error. Let u^0 denote the vector of current centroid displacements. Now recall (8)-(9):

$$\epsilon^x_{ij} = u^0_{ij+1} - u^0_{ij} - h/2(\gamma^x_{ij+1} + \gamma^x_{ij}) \qquad (8)$$

$$\epsilon^y_{ij} = u^0_{i+1j} - u^0_{ij} + \frac{h}{2\sqrt{2}}(\gamma^y_{i+1j} + \gamma^y_{ij} - (\gamma^x_{i+1j} + \gamma^x_{ij})) \qquad (9a).$$

After applying the differential command Δu the adjusted edge error is simply

$$^+\epsilon^x_{ij} = \epsilon^x_{ij} + \Delta u_{ij+1} - \Delta u_{ij},$$

and

$$^+\epsilon^y_{ij} = \epsilon^y_{ij} + \Delta u_{i+1j} - \Delta u_{ij}.$$

Minimizing the vector $^+\epsilon$, $[^+\epsilon^x \;\; ^+\epsilon^y]$ in the least squares sense leads to the problem

$$\min_{\Delta u} |\epsilon + A\Delta u|^2,$$

which is precisely the control law defined in Steps (i)-(iii) above.

This control law is very nearly optimal if the wavefront is locally flat over each subaperture. The proof of this is sketched below. Let $w(x)$ denote the instantaneous wavefront, and let the segmented primary surface be represented by the piecewise linear function $u^0(x)$,

$$u^0(x) = \sum_i \chi(\Delta_i) u^0_i(x)$$

(cf (1)), The objective is to implement a differential command $\Delta u(x)$ of the form

$$\Delta u(x) = \sum \chi(\Delta_i) \Delta u_i(x)$$

with each Δu_i a linear function on Δ_i to minimize the error

$$E(J) = \int_\Lambda |w(x) - u^0(x) + \Delta u(x)|^2 dx, \quad \Lambda = \cup \Delta_i,$$

(cf (2).) Here E denotes the expectation operator, and the requirement is that Δu is measurable with respect to the observed data, that is it must be a function of the tilt and edge sensor measurements. Write $v_i(x) = w(x) - u^0_i(x)$,

$$v_i(x) = \sum_{j=0}^{\infty} \alpha_{ij} T_{ij}(x),$$

where for each i, $\{T_{ij}\}_{j=0}^{\infty}$ is a complete orthronormal system of functions on Λ_i with T_{i0} = piston, T_{i1} = tip, and T_{i2} = tilt. Since $\Lambda u_i(x)$ is linear,

$$\mathrm{AM},(\mathrm{r}) : u_{i0}T_{i0} + u_{i1}T_{i1} + u_{i2}T_{i2}$$

Thus we have

$$E(J) = \sum_{i=1}^{\kappa}\sum_{j=0}^{2} E|\alpha_{ij} - u_{ij}|^2 + \sum_{i=1}^{\kappa}\sum_{j=3}^{\infty} E|\alpha_{ij}|^2,$$

where κ denotes the number of subapertures. Let u denote the vector with components u_{ij}, and let α denote the vector with components α_{ij}, $i = 1, \ldots, \kappa$; $j =$ O, 1, '2. The solution to the optimization problem is to choose u to be the conditional expectation, $\hat{\alpha}$, Of α given the measurements. Assuming $\alpha_{i0} = v(x_i)$, and $[\alpha_i, \alpha_{i2}] = \nabla v(x_i)$, (this is the assumption that the wavefront is locally planar), it can be shown that $\hat{\alpha}$ is the minimum variance solution to the problem

$$\begin{pmatrix} I & 0 \\ \Psi & A \end{pmatrix}\begin{pmatrix} \nabla v \\ v \end{pmatrix} = \begin{pmatrix} y \\ \epsilon \end{pmatrix}.$$

Here y and ϵ are the tilt and edge displacement measurements *before* correction, and Ψ is a matrix that kinematically links the tilt measurement to the edge displacements. The control strategy of first correcting for the tilt, followed by piston correction (or equivalently, minimizing the edge displacements) is t}rc solution obtained by estimating ∇v from the tilt measurement y alone and ignoring the edge sensor measurement altogether. The optimal (i.e., minimum variance) solution couples the tilt and edge sensor measurements at the considerable expense of complicating analysis and inhibiting the development of fast solution techniques. Thus we opt for the suboptimal least squares solution.

Along these same lines we note that although incorporating the data ϵ^s from (9b) is straightforward, this too has a deleterious effect in terms of algorithm deisgn and analysis. The next two sections describe implementation and analysis of the algorithm described in (i)- (iii) above, sans the sensor data ϵ^s. We will show later how to restore this data in an efficient manner.

4. ESTIMATING $u - v$

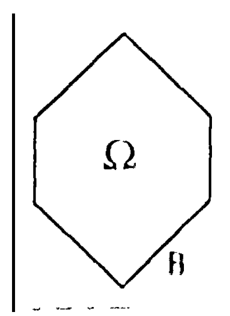
The implementation of the control law outlined above requires solving the least squares problem

$$\min_x |Ax - \epsilon|, \tag{27}$$

where A is the the difference operator defined in (1 O) and ϵ is defined in (24)-(25). On a square grid this can be accomplished via the usc of fast Poisson solver techniques implemented on serial or parallel parallel machines since (27) reduces to the discretized Neumann problem

$$A^T Ax = A^T\epsilon; \quad Ax \cdot n = \epsilon_{boundary} \quad n = \text{boundary normal}$$

Although the SELENE geometry is not square, with a little care the resulting least squares problem can be transformed to a square.



Let Ω denote the region occupied by the SELENE aperture, arid let R denote the region occupied by a circumscribed square. Now A is trivially extended from Ω to R. However, extending the forcing term ϵ

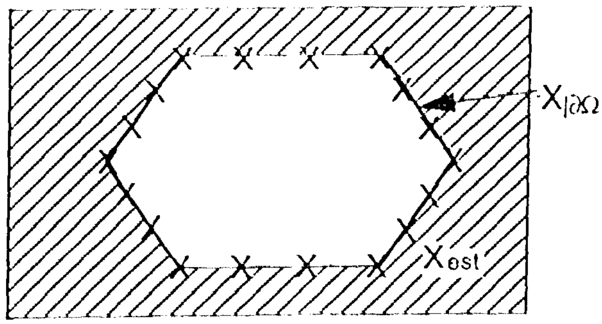
requires a little more consideration. In the deterministic setting there are constraints on how ϵ extends since it is derived from a potential. Specifically, ignoring $O(h^2)$ terms,

$$\sum_{closed \quad contour} \epsilon_{ij} = 0.$$

This constraint can be incorporated in the following way. Suppose we begin with a solution x_{ext} defined on R, then ϵ_{ext} is determined from

$$\epsilon_{ext} = Ax_{ext}.$$

Now if $x_{ext} = x$ in Ω, then $\epsilon_{ext} = \epsilon$ in Ω, and ϵ_{ext} is an extension of ϵ satisfying the velocity constraints. If we can determine the boundary values of x on $\partial\Omega$, call this function $x_{|\partial\Omega}$, then the potential x_{ext} can be defined arbitrarily on $R - \Omega$, and thereby determining an extention ϵ_{ext} to ϵ.



From this discussion we see that the vector field ϵ can be extended (nonuniquely) to the square once an estimate on the boundary is obtained. This approach also works for annular regions since Hartmann sensor data is sufficient to reconstruct these boundary values as well,

To see how this is done let $u_0, u_1, \ldots u_N$ denote the boundary values of x on $\partial\Omega$. The $u_i's$ are related by the difference equation

$$u_{i+1} = u_i + o_i + \eta_i, \quad i = 0, \ldots, N - 1 \tag{28}$$

(o_i : slope measurement, η_i = noise) with the periodic condition

$$u_0 = u_N + o_N + \eta_N. \tag{29}$$

The least squares solution to this problem is obtained by solving the system

$$Ru = f, \quad f = J^T o \tag{30}$$

where

$$R = \begin{pmatrix} 2 & -1 & 0 & 1 \\ -2 & -1 & \ldots & 0 \\ \vdots & \ddots & \vdots & \\ -1 & 0 & \ldots & 2 \end{pmatrix}, \begin{pmatrix} \ldots -1 & & \ldots & 0 \\ & & \ldots & 0 \\ & & \vdots & \\ -1 \quad 2 & & -1 & 1 \end{pmatrix}$$

Now R has null space consisting of the vector $[1 \ldots 1]^T$, corresponding to a piston. However, solutions to this problem are easily obtained by imposing a constraint on u. Once u is obtained, ϵ_{ext} can be defined with just $|\partial\Omega|$ (the number of points on the boundary $\partial\Omega$) adds by taking $x_{ext} = 0$ on $R - \Omega$, and we can proceed to solve the Poisson equation.

This embedding procedure will increase the covariance of the estimate. On an N x N square grid, this covariance grows proportionally to s_N,

$$s_N = \sum_{i,j}^{N} \frac{1}{\lambda_{ij}}; \quad \lambda_{ij} = 4 - 2\cos\frac{\pi i}{N+1} - 2\cos\frac{\pi j}{N+1}.$$

Asymptotically, $s_N = O(N^2 log N)$. To get a handle on how this error grows with the embedding we calculated the ratios s_8/s_4, s_{80}/s_{40}, s_{160}/s_{80}, and s_{320}/s_{160}. Each of these ratios corresponds to the error variance increase resulting from embedding a square region into another square region of twice the size (four times the area). These results are shown below

$$s_8/s_4 = 1.1914, \quad s_{80}/s_{40} = 1.1617, \quad s_{160}/s_{80} = 1.1458, \quad s_{320}/s_{160} = 1.1309.$$

For comparison the asymptotic estimates are

$$s_8/s_4 \approx 1.5, \quad s_{80}/s_{40} \approx 1.1879, \quad s_{160}/s_{80} \approx 1.1582, \quad s_{320}/s_{160} \approx 1.1366.$$

In addition to this analytical analysis, we also did Monte Carlo simulations of the s_8/s_4 case. 500 simulations **were** run and we empirically obtained

$$s_8/s_4 = 1.0698 \text{ (Monte Carlo).}$$

This result is slightly better than anticipated by the analytical estimate. The reason for this is that the embedding procedure *deterministically* adds data. Thus a true minimum variance estimator should very nearly yield a unity ratio for s_8/s_4. The least squares estimator is suboptimal for the embedded problem but should nevertheless produce better results than given by the analytical estimate. As another test we conducted these same simulations without properly embedding the problem. We merely extended the gradient field by using zero values outside of the 4 x 4 square. The results of these Monte Carlo runs were disasterous,

$$s_8/s_4 = 1.3779 \times 10^3 \text{ (Monte Carlo, improper embedding).}$$

This extrapolation approach suggests another fast solution to our problem. SELENE occupies an annular region that can be interpreted as a sequence of rings, say $U^0, \ldots U^N$, where U^0 *is* the outermost ring and U^N *is* the innermost. Note that, U^0 represents the solution on the boundary $\partial\Omega$, and we have already solved this problem in the paragraph above. Clearly we can impose the same solution form for every ring U^i. However, it is necessary to enforce the connectivity between rings. So suppose we **have** obtained estimates **for** $U^0, U^1, \ldots U^{k-1}$ and now seek an estimate for U^k. One difference equation for U^k comes directly from (?8)- (29),

$$U_{i+1}^k = U_i^k + o_i^k + \eta_i^k; \quad U_0^k = U_n^k + o_n^k + \eta_n^k, \quad i = 0, \ldots, n(k) \tag{31}$$

where $n(k)$ denotes the number of segments in the k^{th} ring. With the values $\{U_j^{k-1}\}_{j=1}^{n(k-1)}$, and slope measurements connecting the successive rings, aditional equations are developed for the k^{th} ring. These take on **the** general form

$$U_i^k = T_i(\hat{U}^{k-1}, o^k) \tag{32}$$

where $\hat{U}^{k-1}$ is the estimate of U^{k-1}, o^k is the vector of slope measurements, and T_i is a filtering operator. The bottom line here is that (31) and (X?) can be massaged into a form

$$\begin{bmatrix} J \\ I \end{bmatrix} u = \hat{\epsilon} \tag{33}$$

leading to a " Helmholtz" equation

$$(I + R)U^k = [J^T \; I]\hat{\epsilon} \tag{34}$$

where the right side now contains the estimate $\hat{U}^{k-1}$ as well as the slope measurements.

This looks very much like a filtering operation v,'here information is propagated inwardly, i.e., U^k depends on previous information from the rings $\hat{U}^0, . G.'^{k-1}$ To complete the solution, information should flow in the other direction as well, just as in smoothing. Thus once the entire filtering solution is obtained, 0°,..., $\hat{U}^n$, we should actually repeat the process, going backward from $\hat{U}^N$ to $\hat{U}^0$ to utilize all the information available.

5. ERROR ANALYSIS

Let $\hat{u}(x)$ denote the corrected primary surface. From (3) the error J is given by

$$J = \sum_i \int_{\Lambda_i} |w(x_i) - \hat{u}(x_i) + [\nabla w(x_i) - \nabla \hat{u}(x_i)](x - x_i) + < x - x_i, W(\zeta)(x - x_i) > |^2 dx$$
$$= \sum_i \int_{\Lambda_i} |p_i + \tau_i + q_i|^2 dx,$$

where p_i denotes the piston error ($p_i = w(x_i) - \hat{u}(x_i)$), τ_i denotes the tilt error ($\tau_i = \nabla w(x_i) - \nabla \hat{u}(x_i)$), and q_i denotes the quadratic remainder term. Let E denote the expectation operator. Assuming $E(p_i) = E(\tau_i) = 0$, and that q_i is deterministic we obtain

$$E(J) \leq \sum_i \int_{\Lambda_i} [E(p_i^2) + E(\tau_i^2) + 2E(p_i^2)^{1/2} E(\tau_i^2)^{1/2} + q_i^2] dx.$$

The individual terms in the integrand above will be treated in more detail now. We first analyze the piston error

$$J_{piston} = \sum_i \int_{\Lambda_i} E(p_i^2) dx.$$

Let $v(x_i) = w(x_i) - u(x_i)$. Recall that the control law has the form Au $= u + \hat{v}$. Let a_i^p denote a zero mean random variable representing the actuator piston positioning error. Then

$$p_i = v_i - \hat{v}_i + a_i^p.$$

Assuming a_i^p is independent from the reconstruction error $v_i - \hat{v}_i$, we have

$$E(p_i^2) = E(|v_i - \hat{v}_i|^2) + \sigma_{a^p}^2,$$

where $\sigma_{a^p}^2$ is the variance of the actuator positioning error. Thus,

$$\sum_i \int_{\Lambda_i} E(p_i^2) = \sigma_{a^p}^2 d^2 + \sum_i \int_{\Lambda_i} E(|v_i - \hat{v}_i|^2) dx,$$

where d^2 denotes the aperture area. The second term on the right above is the reconstruction error associated with (26). Write (26) as

$$\epsilon = Av + \eta,$$

where we assume that $E(\eta) = 0$, and $E(\eta\eta^T) = \sigma_{edge}^2 I$. Here σ_{edge}^2 denotes the variance of the edge sensor measurement. Normalizing by the total aperture area d^2 we have the mean square error

$$\frac{1}{d^2} \sum_i \int_{\Lambda_i} E(|v_i - \hat{v}_i|^2) dx = \frac{\sigma_{edge}^2 A(\Lambda_i)}{d^2} \sum_i |v_i - \hat{v}_i|^2$$
$$= \frac{\sigma_{edge}^2 A(\Lambda_i)}{d^2} tr(\Sigma),$$

where $A(\Delta_i)$ denotes the area of the subaperture At and Σ is the covariance matrix of the estimate $\hat{v}$,

$$\Sigma = E((v - \hat{v})^T (v - \hat{v})).$$

Now embed the SELENE aperture into a square aperture with $N \times N$ subapertures. An upper bound for $tr(\Sigma)$ can be developed as4

$$tr(\Sigma) \leq \sum_{i,j}^{N} \frac{1}{\lambda_{ij}}, \quad \lambda_{ij} = 4 - 2\cos\frac{\pi i}{N+1} - 2\cos\frac{\pi j}{N+1}$$

Taking $N = 400$, the sum on the right above is calculated as

$$\sum_{i,j}^{N} \frac{1}{\lambda_{ij}} \approx 1.521 \times 10^5,$$

leading to an rms piston error component

$$RMS_{piston} \leq \sqrt{\frac{2.057 \times 10^5}{M}}\, \sigma_{edge},$$

where M denotes the number of subapertures.

To compute the wavefront error due to tilt error, we will assume that

$$\nabla w(x_i) = \nabla \hat{u}(x_i) = \nu_i,$$

where ν_i is a zero mean random variable with covariance $E(\nu_i\nu_j) = \sigma_\nu^2\delta_{ij}$. Note that ν represents the combined Hartmann sensor error and tilt correction error. Thus

$$\int_{\Delta_i} E(\nu_i^2)dx = \sigma_\nu^2 \int_{\Delta_i} |x - x_i|^2 dx.$$

This integral is computed over the hexagonal region Δ_i as

$$\int_{\Delta_i} E(\nu_i^2)dx = \frac{13}{12}\sigma_\nu^2 h^2 A(\Delta_i).$$

Thus the RMS tilt error is given as

$$RMS_{tilt} = \sqrt{\frac{13}{12}}\sigma_\nu L.$$

Combining these error components we obtain the mean square wavefront error

$$\frac{E(J)}{d^2} \leq \sigma_{a\nu}^2 + \frac{1.521 \times 10^5}{M}\sigma_{edge}^2 + \frac{13}{12}\sigma_\nu^2 h^2 + 2\sigma_\nu h\sqrt{\frac{13}{12}(\sigma_{a\nu}^2 + \frac{1.521 \times 10^5}{M}\sigma_{edge}^2)}$$
$$+ \text{ fitting error.}$$

If we shrink the size of the segments so that $h \to 0$, we observe that N grows linearly with h. And hence because of the asymptotic relationship

$$\sum_{ij}^{N} \frac{1}{\lambda_{ij}} = O(N^2 logN),$$

the rms piston error will grow as $\sqrt{logN}$. Thus the improvement needed in the edge sensors to maintain the same error is rather benign. On the other hand if Hartmann sensors are used for the tilt measurement, the error in this measurement is linearly related to the reciprocal of the segment size. "To see this observe first that for $h < r_0, \sigma_\nu = 3\pi\lambda/16h\sqrt{N_{photons}}$, where $\lambda =$ wavelength, and $N_{photons} =$ number of photons captured over the subaperture[5]. Now $N_{photons}$ is proportional to h_2 Thus σ_ν is proportional to $1/h$.

6. EXTENDING THE ALGORITHM

The algorithm design presented has not included the data e^s (cf(9b)). A straightforward approach for incorporating these measurements is to simply average this data with the other measurements. This approach, although somewhat ad hoc, does not intrude on the algorithmic structure that has been developed, since the modifications are all absorbed in the measurement equation. A more systematic way for incorporating the data is by introducing the difference operator

$$A^* = \begin{bmatrix} A \\ A^s \end{bmatrix},$$

where A is defined as in (10) and A^s is defined

$$A^s u = u_{i+1j} - u_{ij},$$

and measurement vector c^*,

$$c^* = \begin{bmatrix} c \\ c^s \end{bmatrix}.$$

Formally, the analysis proceeds by replacing A and c with A^* and c^*, respectively.

The crux of the implementation problem involving these extended operators and vectors is to solve the new least squares problem

$$\min_x |A^* x - c^*|^2.$$

Using the representations for A^* and c^* above, the solution is obtained via the system

$$[A^T A + A^{sT} A^s]x = A^T c + A^{sT} c^s$$

A promising approach for solving this system that captitalizes on our ability to rapidly invert $A^T A$ is to use a conjugate gradient method[6] with preconditioner $(A^T A)^{-1/2}$. The resulting algorithm is based on iteratively solving systems of the form $A^T A x = b$. We expect such an approach to be efficient and have good convergence properties.

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