NON-LIPSCHITZ DYNAMICS APPROACH TO DISCRETE EVENT SYSTEMS

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ABSTRACT

This paper presents and discusses a mathematical formalism for simulation discrete event dynamics (DED)-a special type of "man-made" systems to serve specific purposes of information processing. The main objective of this work is to demonstrate that the mathematical formalism for DED can be based upon terminal model of Newtonian dynamics which allows one to relax Lipschitz conditions at some discrete points.

A broad class of complex dynamical behaviors can be derived from a simple differential equation [1]:

$$\dot{x} = x^{1/3} \sin \omega t, \omega = cost \tag{1}$$

The solution to Eq. (1) can be presented in a closed form. Indeed, assuming that $x \to 0$ at t =O, one obtains a regular solution:

$$x = \pm\left(\frac{4}{3\omega} \sin^2 \frac{\omega}{2} t\right)^{3/2} \; if \; x \neq 0 \tag{2}$$

and a singular solution (an equilibrium point):

$$x = 0 \tag{3}$$

Clearly, the Lipschitz condition at the equilibrium point x = O fails since

$$\left|\frac{d\dot{x}}{dx}\right| = \frac{1}{3} x^{-\frac{2}{3}} \sin \omega t \to \infty \; at \; x \to 0. \tag{4}$$

As follows from (2.), two different solutions are possible for "almost the same" initial conditions. The fundamental property of this result is that the divergence of these solutions from x = O is characterized by an unbounded parameter, U:

$$\sigma = \lim_{t \to 0} \left[\frac{1}{t} \ln \frac{\left(\frac{4}{3\omega} \sin^2 \frac{\omega}{2} t\right)^{3/2}}{2|x_o|} \right] = \infty, \; |x_o| \to 0 \tag{5}$$

where t_0 is an arbitrarily small (but finite) positive quantity. The rate of divergence (5) can be defined in an arbitrarily small time interval, because the initial infinitesimal distance between the solutions (2) becomes finite during the small interval t_o. One should recall that in the classical case when the Lipschitz condition is satisfied, the distance between two diverging solutions can become finite only at t $\to \infty$ if initially this distance was infinitesimal.

The solution (2) and (3) co-exist at t = O, and that is possible because at this point the Lipschitz condition fails (see Eq. 4),
Since:

$$\frac{\partial \dot{x}}{\partial x} > 0 \ at \ |x| \neq 0, \ t > 0, \tag{6}$$

the singular solution (3) is unstable, and it departs form rest following Eq. (3). This solution has two (positive and negative) branches, and each branch can be chosen with the same probability %. It should be noticed that as a result of(4), the motion of the particle can be initiated by infinitesimal disturbances (that never can occur when the Lipschitz condition is in place: an infinitesimal initial disturbance cannot become finite in finite time).

Strictly speaking, the solution (2) is valid only in the time interval

$$0 \le t \le \frac{2\pi}{\omega}, \tag{7}$$

and at $t = \frac{2\pi}{\omega}$ it coincides with the singular solution (3). For $t > 2\pi/\omega$, Eq. (2) becomes unstable, and the motion repeats itself to the accuracy of the sign in Eq. (2).

Hence, the solution performs oscillations with respect to its zero value in such a way that the positive and negative branches of the solution (2) alternate randomly after each period equal to $2\pi/\omega$.

Let us introduce another variable:

$$\dot{y} = x, \ (y = 0 \ at \ x = 0). \tag{8}$$

After the first ime interval t = $\frac{2\pi}{\omega}$

$$y = \pm \int_0^{\frac{2n}{\omega}} \left(\frac{4}{3\omega} \sin^2 \frac{\omega}{2} t \right) dt = 64(3\omega)^{-5/2} = \pm h \tag{9}$$

After the second time interval t = $\frac{4\pi}{\omega}$

$$y = \pm h \pm h \tag{1o}$$

Obviously, the variable y performs an unrestricted symmetric random walk: after each time period $\tau = 2\pi/\omega$ it changes its value on $\pm h$. The probability $f(y,t)$ is governed by the following difference equation:

$$f(y,t+\frac{2\pi}{\omega}) = \frac{1}{2} f(y - h,t) + \frac{1}{2} f(y + h,t), \int_{-\infty}^{\infty} f(y,t) dy = 1 \tag{12}$$

where h is expressed by Eq. (9).

Eq. (11) defines f as a function of two discrete arguments:

$$y = \pm kh, \ and \ t = l\tau, \ \tau = \frac{2\pi}{\omega}, \ k,l = 0,1,2,...etc. \tag{12}$$

For convenience, we will keep for discrete variable y and t the same notations as for their continuous versions.

By change of the variables:

$$z = \varphi(y), y = \varphi^{-1}(z) \tag{13}$$

one can obtain a stochastic process with a prescribed probability distribution:

$$\psi(z,t) = f\,[\varphi^{-1}(z),t\,]\,|\,\frac{d\varphi^{-1}}{dz} \qquad (14)$$

implemented by the dynamical system (1), *(8),* and (13).

Actually this process represents a piecewise-deterministic Markov process with the correlation time τ. However, by introducing a new variable:

$$\dot{u}(t) = \sum_{q=0}^{n} \alpha_q\, x(t - q\text{-r}), \quad \alpha_q = const \qquad (15)$$

instead of (8), one arrives at a non-Markov stochastic process with the correlation time $(n + 1)\tau$. The deterministic part of the process can be controlled if instead of(8) one applies the following change of variables:

$$\dot{v} = \sum_{q=0}^{n} b_q\, x^{2sq+1}, \quad b_q = const \qquad (16)$$

In particular, the deterministic part of the process can include a pause if, for instance, Eq. (14) is reduced to the following:

$$\dot{u}(t) = x(t) + X(t - \tau) \qquad (17)$$

Let us return to Eq. (1) and assume that it is driven by a vanishingly small input ε:

$$\dot{x} = x^{1/3} \sin \omega t + \varepsilon, \quad \varepsilon \to 0 \qquad (18)$$

From the viewpoint of information processing, this input can be considered as a massage or an event. This massage can be ignored when $\dot{x} \neq 0$, or when $\dot{x} = 0$, but the system is stable, i.e., $x = \pi\omega, 2\pi\omega, ...etc.$ However, it becomes significant during the instants of instability when $\dot{x} = 0$, at $t = 0, \pi/2\omega, ...etc.$ Indeed, at these instants, the solution to (18) would have a choice to be positive or negative if $\varepsilon = 0$, (see Eq. (2)). However, with $\varepsilon \neq 0$

$$sgn\ x = sgn\ \varepsilon \ at\ t = 0\ ,\ \pi/2\omega, ...etc. \qquad (19)$$

i.e. the sign of ε at the critical instances to time (19) uniquely defines the evolution of the dynamical system (18).

Actually the event ε may represent an output of a microsystem which uniquely controls the behavior of the original dynamical system (19).

The probability $f(y, t)$, *is* governed by the following difference equation:

$$f(y,t + \frac{2\pi}{\omega}) = pf(y - h,t) + (1 - p)\, f(y + h,t) \qquad \textbf{(20)}$$

where

$$p = \begin{cases} 1 \ \ if\ sgn\ \varepsilon = 1 \\ 0 \ \ if\ sgn\ \& = -1 \\ \frac{1}{2} \ \ if\ \varepsilon = 0 \end{cases} \qquad (21)$$

Actually, the evolution of the probability distribution in Eq. (20) is represented by rigid shifts of the initial probability distribution $f(y,0)$, unless sgn $\varepsilon = 0$.

The applications of the non-Lipschitz dynamics include stochastic mode] fitting for identification of physical, biological and social systems, simulation of collective behavior, models of neural intelligence [1,2].

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