Moving Frame Formulations of 4-Geometries Having Isometrics

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Abstract

We have previously given well-set exterior differential systems for Ricci-flat 4-geometries as reductions—subbundles—of the bundle of orthonormal frames over 10 dimensional Euclidean (or pseudo-Euclidean) space, lSO(lO).(1) We now understand how to prolong the systems set on 1S0(10) to systems on its tangent bundle TISO(10) and then to its tangent bundle. This in turn makes possible the adjoining of additional invariant forms further restricting the solutions to those having Killing vectors, i.e., isometries. We discuss the geometry of tangent bundles, and report the results of Monte Carlo calculations of the Cartan integer characters of these prolonged exterior differential systems, showing them to be well set for Cauchy-Kowaleski integration, and in the Ricci-flat cases to be causally determined from initial data set on 3-spaces. Similar results are reported for Ricci-flat geometries with two isometrics.

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L The Structure of TG

When the intrinsic structure of a group manifold G is expressed by Cartan-Maurer equations for its left (right)-invariant basis l-forms η^a (a = 1.. . .n, the dimension of the manifold) we have the identities

$$d\eta^a + \frac{1}{2} C^a_{bc}\, \eta^b \wedge \eta^c = 0. \tag{1}$$

C^a_{bc} are the structure constants, satisfying Jacobi identities, so the 3-form exterior derivatives of (1) are quickly shown to vanish, modulo (l). The set (1) is closed.

To(1) may be added an additional set in n further basis 1-forms $\bar{\eta}^a$:

$$d\bar{\eta}^a + \frac{1}{2} C^a_{bc} \left(\bar{\eta}^b \wedge \eta^c + \eta^b \wedge \bar{\eta}^c\right) = 0. \tag{2}$$

Closure is again immediate, modulo (1) and (2), This introduction of additional bases has been called co-adjoint prolongation[2] as clearly it is related to the structure of the linear adjoint representation of G. (1) and (2) together are the Cartan Maurer equations of a Lie group with 2n invariant bases η^a and η^a. To identify this larger group geometrically we will consider the dual basis of left (right) -invariant vector fields on G, say V_a where $V_a \rfloor \eta^b = \delta^b_a$. They satisfy

$$[V_a, V_b] = C^c_{ab} V_c \quad . \tag{3}$$

We use the concept of lifting in a geometric object bundle[2]. Roughly, a geometric object is a (multi-component) field-cross section of a bundle-that has also a well defined Lie derivative (and so a covariance property under diffeomorphisms). (Tensor fields are linear geometric objects). This is equivalent to the statement that vector fields—diffeomorphism generators—in the base space of such a bundle can be lifted, into vector fields that operate on the bundle manifold, preserving their Lie product relations. Vector fields on a manifold M are also geometric object fields, cross sections of the tangent bundle TM. This dual rôle yields two intrinsic lifts of them into TM(3). In terms of a local coordinate frame xi on M, introduce coordinates vi in the fibers of TM such that at xi a tangent vector $V = V^i \frac{\partial}{\partial x^i}$ is the point $(x^i, v^i = V^i)$ in TM. The covariant (or complete) lift of a field $V = V^i(x)\, \frac{\partial}{\partial x^i}$ (a cross section of TM) is

$$ {}^{c}V \equiv V^i(x)\frac{\partial}{\partial x^i} + V^i_{,j}(x)\, v^j \frac{\partial}{\partial v^i} \tag{4}$$

(a vector field throughout TM). The second lift is the so-called <u>vertical</u> lift

$$ {}^{v}V \equiv V^i(x)\frac{\partial}{\partial v^i} \quad . \tag{5}$$

The complete lift expresses covariance: three diffeomorphism generators U, V, W in M that satisfy [U, v] = W there, have lifts in TM that, from (4), satisfy

$$\left[{}^{c}U,\ {}^{c}V\right] = {}^{c}W \tag{6}$$

From (4) and (5) we further calculate

$$\left[{}^{c}U,\ {}^{v}V\right] = {}^{v}W = \left[{}^{v}U,\ {}^{c}V\right] \tag{7}$$

which shows the vertical lift operation also to be intrinsic, and finally

$$\left[{}^{v}U,\ {}^{v}V\right] = 0. \tag{8}$$

If M is a group manifold G, applying these two lifts to the basis vectors in (3)

$$\begin{aligned} &[{}^{c}V_a,\ {}^{c}V_b] = C^c_{ab}\ {}^{c}V_c \\ &[{}'V_A,\ {}^{v}V_b] = C^c_{ab}\ {}'V_c \\ &[{}^{v}V_a,\ {}^{v}V_b] = 0 \quad . \end{aligned} \tag{9}$$

These are immediately seen to be Lie-algebraically-dual to the co-adjoint prolongation of G, viz. (1) and (2), so we have shown (1) and (2) to be the Cartan-Maurer structure equations of TG. The direct lift (pull back) of the η^a will still be denoted η^a; the $\bar{\eta}^a$ are dual to vertically lifted basis vectors. η^a and $\bar{\eta}^a$ thus are a canonical basis on TG. An n-dimensional cross section of TG is, we repeat, a vector field on G.

II. G-prolongation

The same formal prolongation leading from (1) to (2) has been used by Gürses[4] to discuss invariance of the spinor equations for Ricci-flat 4-geometries. Spinor, or d yadic, or tetrad formalisms for the Einstein field equations can all be seen as the setting of exterior differential systems on suitable flat frame bundles (which are in fact group manifolds), reducing them to non-flat bundles over 4-dimensional base manifolds(l). By considering variations of the spinor equations Gürses essentially doubles the number of variables (although this is not clear when variations of higher order forms are introduced without first expressing these forms in terms of basis 1-forms). He interpreted this as leading to the invariance group of the field equations; our discussion above rather agrees with later comment[5] that one thus finds general diffeomorphism generators-arbitrary vector fields on any solution manifold. Nevertheless Gürses prolongation technique on differential systems on bundies is a significant generalization of co-adjoint prolongation of group manifolds. We will denote it G-prolongation. Our detailed calculations of specific exterior differential systems, some of which are reported below, all show G-prolongation to be consistent and well set. We have realized that it allows us to formulate invariant differential systems for Ricci-flat space times having symmetries —Killing vectors–situations which we have not previously been able easily to discuss in terms of Cartanian moving frame systems. Instead of forcing auxiliary vectors into the Cartan formalism, we now just double (and redouble) the basis forms, and then specialize the solutions by adding extra forms to the G-prolonged exterior differential systems.

111. Prolonged Systems with One Vector Field

We have previously[6][1] motivated and described the key diagnostic calculations for an exterior differential system in n dimensions: the identification of its Cauchy characteristic vectors and the calculation of its Cartan characteristic integerss = ($s_0, s_1, s_2 \ldots s_{g-1}$). g is the "genus" of the system, the dimension of its maximal, generic, integral manifolds (solutions of the equivalent set of first order partial differential equations). These integral manifolds are fiber bundles, the fibers being generated by the Cauchy characteristics. The integers s are calculated from the ranks of a nested series of linear homogeneous equations. If $\sum_{0}^{g-1} s_i = n - g$ the system is well set, and generic solutions are found by systematic Cauchy-Kowaleski integrations from data set on a nested sequence of submanifolds of dimension 1, 2, ... If $s_i=0$ for $i \geq q$, the essential C-K data are s_{q-1} functions set on q- 1 -dimensional submanifolds; when the solution is a bundle over a q-dimensional base, we call the system causal[6].

The calculation ofs is highly non-trivial, and in the mathematics literature results for exterior systems generated by forms of degree >2 are hardly to be found. Many classic problems of embedding of submanifolds are discussed with Cartanian methods by Jensen and Griffiths[7], but when higher rank forms are adjoined, for example to restrict solutions to be Ricci-flat, analytical calculation ofs becomes almost impossible. We have instead developed some symbolic manipulation programs to handle large numbers of indexed basis forms, and a Monte Carlo approach to sequentially calculate the ranks giving the integers in the sets. An example of this has now been explicitly explained and the computer output displayed[8]. The results reported below have all been calculated in this way.

We begin with the group G=ISO(10), the orthogonal frame bundle over 10 dimensional Euclidean space. In terms of 55 canonical basis forms co$^{\mu}$, $\omega^{\mu}_{\nu} = -\omega^{\mu}_{\nu}$, μ = 1...10 (the signs can be adjusted afterwards for different signatures), satisfying the usual structure relations[1], the exterior differential system for immersed 4-geometries is the ideal generated by the invariant forms[1]

$$\begin{gathered} \omega^{A} \\ \omega^{A}_{i} \wedge \omega^{i} \end{gathered} \qquad \text{(1o)}$$

where i = 1··4, A = 5-10. There are 21 Cauchy characteristic vectors (dual to ω^{i}_{j} and ω^{A}_{B}, which do not occur in (10)); the Cartan characteristic integers ares = (6, 6, 6,6,6, 0,.-") and g = 25. $\sum_{0}^{g-1} s_j$ = 30 = 55-25 so (10) is a well-set Cauchy-Kowaleski set of first order p.d.e.'s (when expanded in any 55 dimensional set of coordinates on 1S0(1 O)). Solutions are 25 dimensional, fibered over a 4 manifold. The fibers are each O(4) ⊗ O(6). A cross section is a realization of an orthogonal frame field, plus 15 auxiliary gauge fields spanning O(6). The Riemann tensor is coded in the curvature 2-forms $\omega^{I}_{A} \wedge \omega^{J}_{A}$.

Now G-prolong both 1S0(10) and the generating set (10). We have an exterior system on TISO(10), dim 110, generated by

$$\begin{gathered} \omega^{A} \\ \bar{\omega}^{A} \\ \omega^{A}_{i} \wedge \omega^{i} \\ \bar{\omega}^{A}_{i} \wedge \omega^{i} + \omega^{A}_{i} \wedge \bar{\omega}^{i} \end{gathered} \qquad \text{(11)}$$

Monte Carlo calculation “gives s = { 12, 12, 12, 12, 12, 0,”.), solutions are g = 50 dimensional, 42 dimensional fibers over 8 dimensional bases. Evidently, solutions are pairs: immersed 4-geometries with tangent bundles over them.

If four 3-forms $\omega^i_A \wedge \omega^j_A \wedge \omega^k \varepsilon_{ijkl}$, coding for the condition that the Ricci tensor induced *on* the sub-manifold vanishes, are added to (10) the characters become(1) s = {6,6, 10, 8, 0,.-}, g = 25. It is noteworthy that now *s4* = O: generic solutions arise from eight functions set on 3-spaces, demonstrating the causal, dynamical property of the Einstein field equations. The G-prolonged system on TISO(10) has generators

$$
\begin{gathered}
\omega^A \\
\bar{\omega}^A \\
\omega^A_i \wedge \omega^i \\
\bar{\omega}^A_i \wedge \omega^i + \omega^A_i \wedge \bar{\omega}^i \\
\omega^i_A \wedge \omega^j_A \wedge \omega^k \epsilon_{ijkl} \\
(2\bar{\omega}^i_A \wedge \omega^j_A \wedge \omega^k + \omega^i_A \wedge \omega^j_A \wedge \bar{\omega}^k)\epsilon_{ijkl}
\end{gathered} \tag{12}
$$

and we find by Monte Carlo calculation s = (12, 12,20, 16, 0,···} g = 50. We interpret this again as allowing an arbitrary vector field in any solution. We have calculated a number of other sets with the same result, and as explained above believe this G-prolongation to be a neat technique generalizing co-adjoint prolongation of Lie groups.

The new application we have made, then, is to specialize to a Killing vector field, by further adjoining to either (11) or (12) the generating forms

$$
\begin{gathered}
\bar{\omega}^i \\
\bar{\omega}^i_j \\
\bar{\omega}^i_A \wedge \omega^j_A + \omega^i_A \wedge \bar{\omega}^j_A
\end{gathered} \qquad 03)
$$

which are closed modulo themselves and either (11) or (12) and which require the variations of the ~~frame~~ connection field and the curvature 2-forms $\omega^i_A \wedge \omega^j_A$ to vanish. Note that the immersion is made specific by further requiring preservation of the lines of curvature (that is, we also include the variations $\bar{\omega}^i$ in the ideal). For general 4-metrics using (1 1), we calculate s = {22, 18, 14, 10,6, 0,···). The most interesting result is for Ricci-flat 4-geometries having one Killing vector; the combined ideal (12) plus (13) yields s = {22, 18, 18, 12, 0,-.) g = 40. There are 36 Cauchy characteristics, six corresponding to frame covariance (co; not present) and 30 gauge fields

$\left(\omega^{A}_{B}\right.$ and $\bar{\omega}^{A}_{B}$ not present$\left.\right)$, The invariant differential system for these space times is well set and causal: the generic solution depends on 12 initial value functions ~~over~~ on a 3 space.

IV, Prolongation with Two Vector Fields

We have considered further prolongation to the second tangent bundle. This appears to be at the practical limit of our present computational ability, but it makes possible an invariant two Killing vector formulation that should make contact with many previously known results: soliton hierarchies for the Ernst equation, for example.

The structure of TTG is interesting in itself[9][10]. Now there are four lifts of vector fields from G—we label them cc, CV, vc and vv in an obvious notation and find by iterating (9)

$$
\begin{aligned}
[{}^{cc}V_a, {}^{cc}V_b] &= C^{c}_{ab}\, {}^{cc}V_c \\
[{}^{cc}V_a, {}^{cv}V_b] &= C^{c}_{ab}\, {}^{cv}V_c \\
[{}^{cc}V_a, {}^{vc}V_b] &= C^{c}_{ab}\, {}^{vc}V_c \\
[{}^{cc}V_a, {}^{vv}V_b] &= C^{c}_{ab}\, {}^{vv}V_c \\
[{}^{cv}V_a, {}^{vc}V_b] &= C^{c}_{ab}\, {}^{vv}V_c \\
[{}^{vv}V_a, {}^{vc}V_b] &= [{}^{vv}V_a, {}^{cv}V_b] = 0 \\
[{}^{vc}V_a, {}^{vc}V_b] &= [{}^{vv}V_a, {}^{vv}V_b] = [{}^{cv}V_a, {}^{cv}V_b] = 0 .
\end{aligned}
\tag{14}
$$

Dual forms satisfy a G-prolongation of (1) and (2):

$$
\begin{aligned}
&d\omega^a + \tfrac{1}{2} C^{a}_{bc}\, \omega^b \wedge \omega^c = 0 \\
&d\bar{\omega}^a + \tfrac{1}{2} C^{a}_{bc} (\bar{\omega}^b \wedge \omega^c + \omega^b \wedge \bar{\omega}^c) = 0 \\
&d\tilde{\omega}^a + \tfrac{1}{2} C^{a}_{bc} (\tilde{\omega}^b \wedge \omega^c + \omega^b \wedge \tilde{\omega}^c) = 0 \\
&d\mathring{\omega}^a + \tfrac{1}{2} C^{a}_{bc} (\mathring{\omega}^b \wedge \omega^c + \bar{\omega}^b \wedge \tilde{\omega}^c + \tilde{\omega}^b \wedge \bar{\omega}^c + \omega^b \wedge \mathring{\omega}^c) = 0
\end{aligned}
\tag{15}
$$

TTG is a nonlinear geometric object bundle with 3n-dimensional fibers, and so has a Lie derivative for fields that are cross sections[2]. Such fields—cross sections of iterated tangent bundles—have been called <u>sectors</u> by J. E. White. They are carefully discussed, and their (non-linear) Lie derivatives derived, in his mathematical monograph[10]. Points in the fibers of the second tangent bundle of M are local elements of 2-manifolds in M; as a cross section of ‘I’M is a (calibrated) n- 1 dimensional congruence of lines in n-space, so is a cross section of TTM a realization of the base equipped with an n-2 dimensional congruence of 2-manifolds. If such a construction were attempted with two copies of cross sections of TG—linear vector fields-— we would have to adjoin further conditions for them to be 2-forming. As a cross section of TTG, this closure is automatic,

Prolongation to formulate geometries having two 2-forming Killing vector fields then is straightforward, However as the number of dimensions doubles again, we find our Monte Carlo program unable to give a reliable set of characters for them= 4 case.* For the simpler case of a 3 dimensional Riemannian geometry immersed in 6 dimensional flat space, i = 1,2,3 and A = 4,5, 6, the dimension of 1S0(6) is 21 and the problem with two isometrics is set on its second tangent bundle, TTISO(6), dim 84. This we have analyzed successfully.

To be explicit, the complete exterior differential system we set on TTISO(6), dim 84, is 30 1-forms:

$$\omega^A,\ \bar\omega^A,\ \tilde\omega^A,\ \mathring{\omega}^A,\ \bar\omega^i,\ \tilde\omega^i,\ \mathring{\omega}^i,\ \bar\omega_j^i,\ \tilde\omega_j^i,\ \mathring{\omega}_j^i, \qquad (16)\ \cancel{(17)}$$

and 28 2-forms:

$$\omega_i^A\wedge\omega^i,\ \bar\omega_i^A\wedge\omega^i,\ \tilde\omega_i^A\wedge\omega^i,\ \mathring{\omega}_i^A\wedge\omega^i,\ \bar\omega_A^i\wedge\omega_A^j+\omega_A^i\wedge\bar\omega_A^j,$$
$$\tilde\omega_A^i\wedge\omega_A^j+\omega_A^i\wedge\tilde\omega_A^j,\ \mathring{\omega}_A^i\wedge\omega_A^j+\bar\omega_A^i\wedge\tilde\omega_A^j+\tilde\omega_A^i\wedge\bar\omega_A^j+\omega_A^i\wedge\mathring{\omega}_A^j, \qquad (17)\ \cancel{(18)}$$

and to this we can add three 2-forms for Ricci-flatness. The results of the Monte Carlo computations are that the systems are well set, and in the second case causal (2+1 gravity with invariant lines of curvatures):

n = 84, s= {30, 21, 12, 3, 0,···} g= 18	general 3-geometries with 2 isometrics
n = 84, s = (30, 24, 12, 0,···} g= 18	flat 3 geometries with 2 isometrics

Three dimensional cross sections of solutions are 3 connection form fields and 12 gauge fields (respectively dual to $\omega^i_j, \omega^A_B, \bar\omega^A_B, \tilde\omega^A_B, \mathring{\omega}^A_B$).

* We suspect the characters for general 4-spaces with 2 isometrics to bes = {54, 42,30, 18,6, 0,...), g =70, 66 dimensional fibers over 4 space, and that with 4 additional 3-forms for Ricci-flatness they become causal: s = (54, 42, 34, 20, O,...),

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