ESTIMATING THE MODIFIED ALLAN VARIANCE

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The third-difference approach to modified Allan variance (MVAR) leads to a tractable formula for a measure of MVAR estimator confidence, the equivalent degrees of freedom (edf), in the presence of power-law phase noise. The effect of estimation stride on edf is tabulated. A simple approximation for edf is given, and its errors are tabulated. A theorem allowing conservative estimates of edf in the presence of compound noise processes is given.

Introduction

The ingredients for this work were presented three years ago at this Symposium. The first ingredient, a paper by the present author [6], shows how the labor of computing modified Allan variance (MVAR) estimates can be reduced by expressing MVAR in terms of third differences of the cumulative sum of time residuals. This approach shows that an MVAR estimate is hardly more difficult to compute than a conventional Allan variance (AVAR) estimate. A review of the method is given below. The second ingredient is a paper by Kasdin and Walter [10] on simulating a class of discrete-time power-law noises. In a subsequent paper [12], Walter exploits these noise models to derive a formula for the variance of a fully overlapped MVAR estimator. Combined with a formula for the estimator mean (MVAR itself), this formula can be used for computing an estimator confidence measure, the <u>equivalent degrees of freedom</u> (edf, defined below). In turn, edf can be used for assigning confidence intervals.

Walter's expression is difficult to evaluate. Happily, the combination of Walter's models with the third-difference approach has led to another formula for edf, mathematically equivalent to Walter's formula, but easier to evaluate because it has fewer summation terms. This formula is given below, together with additional results as follows.

- An assessment of the dependence of the edf of an MVAR estimator on its estimation period τ_1, defined as the time interval by which the summands of the estimator are shifted. It turns out that a wide range of choices of τ_1 gives essentially the same edf. The user can choose τ_1 from considerations of convenience and computational effort.

- A simple approximation formula for edf, with coefficients drawn from a brief lookup table. Most users will not need the exact edf formula.
- A theorem that allows one to calculate conservative values of estimator edf in the presence of a polynomial phase noise spectrum, i.e. , linear combinations of power laws with unknown coefficients. This theorem is also valid for AVAR estimators, but is more useful for MVAR estimators because their edf varies less with power-law noise exponent.

The most critical assumption underlying these results is a negligible rate of linear frequency drift, or a drift rate that is known a priori; in this case, it can be removed from the data.

This paper mainly gives results; a longer paper with more derivations [7] has been submitted elsewhere.

MVAR and Its Estimators

Third-Difference Formulation

Let $x_1, x_2, \ldots,$ with sample period τ_0, be a sequence of time residuals obtained from a comparison of clocks or from a phase comparison of two frequency sources. The conventional Allan variance for an averaging time $\tau = m\tau_0$ is defined by

$$\sigma_y^2(\tau) = \frac{1}{2\tau^2} E[\Delta_m^2 x_n]^2, \tag{1}$$

where E denotes mathematical expectation (ensemble average) , and Δ_m is the backward difference operator with step m, that is,

$$\Delta_m f_n = f_n - f_{n-m},$$

$$\Delta_m^2 f_n = f_n - 2f_{n-m} + f_{n-2m},$$

for any sequence f_n.

For the modified Allan variance, define the moving averages of x_n by

$$\bar{x}_n(m) = \frac{1}{m}\sum_{j=0}^{m-1} x_{n-j}.$$

'l'he conventional definition of MVAR is

$$\text{mod}\,\sigma_y^2(\tau) = \frac{1}{2\tau^2}\, E\left[\Delta_m^2 \bar{x}_n(m)\right]^2. \qquad (2)$$

The third-difference formulation of MVAR uses an auxiliary sequence w_n of cumulative sums of x_n, defined by

$$w_0 = 0, \quad w_n = \sum_{j=1}^{n} x_j \qquad (3)$$

This sequence can be generated from the recurrence $w_n = w_{n-1} + x_n$. Observe that

$$\bar{x}_n(m) = \frac{1}{m}\Delta_m w_n, \quad n \geq m.$$

When this is substituted into (2), the difference operators multiply to give

$$\text{mod}\,\sigma_y^2(\tau) = \frac{1}{2\tau^2 m^2}\, E\left[\Delta_m^3 w_n\right]^2$$

$$= \frac{1}{2\tau^2 m^2}\, E\left[w_n - 3w_{n-m} + 3w_{n-2m} - w_{n-3m}\right]^2. \qquad (4)$$

This is the third-difference form of MVAR. The advantage of (4) over (2) is that it expresses MVAR in terms of four values of w_n instead of 3m values of x_n.

MVAR Estimator with Stride

To estimate MVAR with limited data, we replace the E operator in (4) by a finite average over n. For such a time average, we have to decide how much to increase n from one term to the next. This increase, denoted here by m_1, is called the estimation stride. The corresponding time shift $\tau_1 = m_1\tau_0$ is called the estimation period. When computing AVAR from (1), it is customary to use $m_1 = 1$, called "full overlap", or $m_1 = m$, called "τ overlap". The effect of these choices on AVAR estimator confidence has previously been computed ([5] and references therein). In the context of MVAR, the overlap formalism becomes awkward, and is replaced here by the stride formalism. The existing literature on MVAR ([1], for example) customarily assumes" a stride of 1 (with good reason, as we shall see later). Here, we shall allow m_1 to vary between 1 and m, and investigate the effect on the confidence of the resulting estimator.

Suppose that N time residuals $x_1, x_2, \ldots, x_N$ are available. From these come N+1 cumulative sums $w_0, w_1, \ldots, w_N$, and N-3m+1 samples of $\Delta_m^3 w_n$, $3m \le n \le N$. Let M be the number of samples of $\Delta_m^3 w_n$ that are separated by the stride m_1. Then

$$M = \mathrm{int}\left[\frac{N - 3m + m_1}{m_1}\right], \tag{5}$$

where int(x) is the greatest integer that is $\le x$. The MVAR estimator to be studied is given by

$$V = \frac{1}{2\tau^2 m^2 M} \sum_{k=0}^{M-1} \left[\Delta_m^3 w_{3m+km_1}\right]^2 . \tag{6}$$

Equivalent Degrees of Freedom

One measure of the statistical confidence of an estimator X is its equivalent degrees of freedom (edf), defined by

$$\mathrm{edf}\ X = \frac{2(EX)^2}{\mathrm{var}\ X}. \tag{7}$$

Higher values of edf mean that the distribution of X is more concentrated about its mean. If X is distributed as a constant multiple of a chi-squared random variable with ν degrees of freedom, then edf X = ν. Even if X does not have such a distribution, edf X can still serve as a convenient dimensionless measure of the confidence of X as an estimator of its mean. In this case, edf X need not be an integer. I take this point of view with regard to V, not having studied the nature of its distribution under the noise models discussed below. In frequency-stability analysis, it is customary to assume that estimators of AVAR or MVAR obey an approximate chi-squared law, and, on this basis, to construct confidence intervals for AVAR or MVAR [9][15) from levels of the appropriate chi-squared distribution function.

Noise Models

The statistical properties of V, its edf in particular, depend on the random process chosen to model the time residuals x_n. The classical continuous-time spectral model for phase or time deviations is a linear combination of power laws:

$$S_x^+(f) = \sum_{\beta=-4}^{0} g_\beta f^\beta, \tag{8}$$

whose components, for β = 0,-1,-2,-3,-4, are called white phase, flicker phase, white frequency, flicker frequency, and random-walk frequency. (The plus sign indicates one-sided spectral density.) It is understood that there is some high-frequency cutoff , the "hardware bandwidth", and that the power-law components of (8) might only behave asymptotically like f^β as $f \to 0$. Bernier [2] studied the behavior of MVAR for each of these spectral components, tackling the complex interaction among the hardware bandwidth B, the sample period τ_0, and the averaging time τ. Here, we follow Walter [12] in using explicit discrete-time power-law models for the samples X_n of the time residual process. These are the so-called fractional-difference Processes [3][8], which have one-sided spectral densities proportional to

$$S_x^+(f) = 2[2\sin(\pi f \tau_0)]^\beta, \quad f < \frac{1}{2\tau_0}. \tag{9}$$

Nonintegral values of β are allowed.

There are two reasons for using these models here. First, the abovementioned complications of sampling the continous-time models are avoided. Second, the models fit perfectly into the MVAR third-difference framework. In particular, the sequence w_n defined by (3) is also a fractional-difference process with exponent β-2, that is,

$$S_w^+(f) = 2[2\sin(\pi f \tau_0)]^{\beta-2}. \tag{10}$$

Now , since MVAR has been given in terms of w_n, there is no need to use x_n in the theory.

Generalized Autocovariance

The frequency-domain description (10) of the model for w_n has an equivalent time-domain description, called the generalized autocovariance (GACV) and denoted by $R_w(n)$, where n runs through all integers, positive and negative. The concept of autocovariance (ACV) as a function of one time variable applies to stationary processes only. With some care, though, it can be extended to certain nonstationary processes in such a way that their covariance properties can be described in terms of a function, the GACV, that still depends on one time variable. Although the GACV cannot be regarded as a covariance function in the usual sense, it can be used like one under certain restrictions.

Because the GACV $R_w(n)$ plays a central role in the formula for edf V given below, we give this function here for all the required values of β, namely $-4 \leq \beta \leq 0$. Bear in mind that the noise-type label applies to X_n, a power-law process with exponent β, while $R_w(n)$ applies to a power-law process w_n with exponent $\beta-2$. For the flicker noises we need an auxiliary sequence L_n, a discrete version of the logarithm, defined by

$$L_0 = 0, \quad L_n = \sum_{j=1}^{n} \frac{1}{j - 1/2}.$$

Following are the required GACV formulas.

$\beta = 0$; white phase

$$R_w(n) = \frac{-|n|}{2\tau_0}$$

$\beta = -1$; flicker phase

$$R_w(n) = \frac{-1}{2\pi\tau_0} \left[\frac{1}{4} - n^2\right] L_{|n|}$$

$\beta = -2$; white frequency

$$R_w(n) = \frac{-|n|(1 - n^2)}{12\tau_0}$$

$\beta = -3$; flicker frequency

$$R_w(n) = \frac{-1}{24\pi\tau_0} \left[\frac{1}{4} - n^2\right] \left[\frac{9}{4} - n^2\right] L_{|n|}$$

$\beta = -4$; random-walk frequency

$$R_w(n) = \frac{-|n|(1 - n^2)(4 - n^2)}{240\tau_0}$$

β nonintegral

$$R_w(n) = \frac{-\Gamma(1-\beta/2+n)}{2\tau_0 \cos(\pi\beta/2)\ \Gamma(2-\beta)\ \Gamma(\beta/2+n)}$$

The formula" for nonintegral β is equivalent to the form used by Kasdin and Walter [10] and by Walter [12], but for the GACV of x_n, not of w_n.

Additional Mathematical Assumptions

For technical correctness, it is assumed that the the time residuals X_n have stationary, Gaussian, mean-zero second increments. Assuming that these increments have zero mean is the same as assuming that the frequency drift rate is zero.

Results

MVAR Estimator edf: Exact Formula

In the estimator defined by (5) and (6), recall that the averaging time is $m\tau_0$, and that the estimation period is $m_1\tau_0$. For nonnegative integers n, let

$$\begin{aligned} R_n = & -R_w(n-3m) + 6R_w(n-2m) \\ & - 15R_w(n-m) + 20R_w(n) - 15R_w(n+m) \\ & + 6R_w(n+2m) - R_w(n+3m). \end{aligned} \tag{11}$$

In other words, $R_n = -\delta_m^6 R_w(n)$, where δ_m^6 is the sixth central difference operator with step m. Actually, R_n is just the ordinary ACV of the stationary process $\Delta_m^3 w_n$. Let

$$\rho_n = \frac{R_n}{R_0},$$

the corresponding autocorrelation sequence. The formula for edf V is given by

$$\frac{1}{\text{edf V}} = \frac{1}{M}\left[1 + 2\sum_{k=1}^{M-1}\left(1 - \frac{k}{M}\right)\rho_{km_1}^2\right]. \tag{12}$$

This formula is mathematically equivalent to Walter's formula for var V ([12], eq (32)), but requires less computation. Evaluation of (12) requires 7M evaluations of $R_w(n)$. Walter's formula, which is given only for $m_1 = 1$, is a double sum requiring $5(2m-1)(2M-1)$ evaluations of the GACV of X_n. This shows the advantage of the third-difference approach, which derives MVAR estimator summands from four values of w_n instead of from 3m values of x_n.

In connection with a recent conference paper [15], tables of edf V for $m_1 = 1$ and integral β were generated by the method given here, by Walter's method, and by Monte Carlo simulation. The two theoretical methods agreed within 0.1 percent; the simulations agreed with the theoretical results within a few percent.

A note on computation: The ACV R_n tends to zero as $n \to \infty$, yet is obtained from differences of $R_w(n)$, which tends to ∞ with n. Clearly, one should use double precision for evaluating (11). Even so, the computed values of R_n can deteriorate for large n, especially for nonintegral β, where $R_w(n)$ involves Γ functions. I was able to cure this problem by replacing the upper limit M-1 of the summation in (12) by K-1, where $K = \min(M, 10m/m_1)$. (In all actual computations, m/m_1 is assumed to be an integer.)

Effect of Estimation Period

From here on, we assume that the estimation period divides evenly into the averaging time, that is, we have

$$\frac{\tau}{\tau_1} = \frac{m}{m_1} = r,$$

where r is an integer. Under this assumption, (12) was used to generate tables of edf V for combinations of N, m, and m_1. For each combination, the number M of estimator summands is calculated from (5), and the parameter p is defined by

$$p = \frac{M}{r} = \frac{M\tau_1}{\tau}. \tag{13}$$

A selection of edf values is shown in Table 1 for integral values of β. Values for half-integral β were also computed, but are not shown; as expected, they interpolate the given values. For now, ignore the "%" rows, and observe how edf depends on r (or m_1) for N = 1024, m fixed. For each β, and for $m \geq 4$, it is clear that any value of r between 4 and m gives a value of edf that is nearly maximal for that m and β. If $m < 4$, then we should take $\tau_1 = \tau_0$ ($m_1 = 1$, $r = m$). For $\beta \geq -2$, an estimation period of τ ($m_1 = m$, $r = 1$) gives inferior results. Here is an empirical result that summarizes the observations.

Assume an averaging time τ at most 1/4th the duration of the time-deviation record. For each power law between white phase and random-walk frequency, any estimation period τ_1 between τ_0 and $\max(\tau_0, \tau/4)$ that divides evenly into τ gives an MVAR estimator V whose edf is within 8 percent of the maximal value for τ.

Table 1 shows that the variation of edf V with r is greatest for white phase ($\beta = 0$). Also, we see that p itself is a rough estimate of edf V, especially for m_1 in the recommended range $1 \leq m_1 \leq \max(1, m/4)$.

The choice of estimation period τ_1 might depend on a tradeoff between convenience and computational effort. For small data sets that are held entirely in memory, the minimal choice $m_1 = 1$ is convenient, and the computational cost is probably negligible. For larger data sets that. are read sequentially from a file, the maximal choice $m_1 = m/4$ allows sequential accumulation of MVAR sums from the stream of w_n with moderate use of memory. As an example, take $m = 32$, $m_1 = 8$. To update the sum of squares of $\Delta^3_{32} w_n$ at every eighth sample of w_n, the program has to remember the previous 12 values of w_{8j}. Alternatively, if there are many thousand data points, one can simply use $m_1 = m$ to accumulate sums of squares of third differences for smaller values of m, while collecting a global buffer of w_n subsampled by some factor m_2. After all the data are read, the buffer is used for calculating MVAR estimates with $m_1 = m_2$, $m = rm_1$ for various r.

MVAR Estimator edf: Approximate Formula

Because the power-law models are only an approximate fit to actual phase noise, the precision of the theoretical values of edf V in Table 1, four significant figures, is meaningless for a user who needs to construct error bars for MVAR measurements. Therefore, the following simple approximation is offered as an empirical result.

Assume power-law phase noise with exponent β between -4 (random-walk frequency) and 0 (white phase), at least 16 time-residual points, an averaging time τ at most 1/5th the duration of the measurement, and an estimation period τ_1 between τ_0 and $\max(\tau_0, \tau/4)$ that divides evenly into r. In our notation, $N \geq 16$, $m \leq N/5$, and $m = rm_1$, where r is an integer between $\min(m,4)$ and m. For the estimator V defined by (6), we have

$$\text{edf } V \approx \frac{a_0 p}{1 - \frac{a_1}{p}}, \qquad (14)$$

where $p = M/r$, M is given by (5), and the coefficients a_0, a_1, as functions of m and β, are drawn from Table. 2.

The relative error of this approximation is observed to be at most ±11.1 percent.

Each "%" row in Table 1 shows the percentage errors of (14) for the row above. The table entries were chosen to represent the full range of. observed errors. This approximation holds only under the above restrictions on data set size and averaging time. For example, if m = N/4 then the exact edf formula (12) must be used.

This approximation was derived from two rigorous lower bound formulas, one for edf V, the other for the edf of a continuous-time analog of V. The choice between these two bounds as approximations was made partly by insight, partly by trial and error.

Compound Noise Spectra

The foregoing results assume a power-law phase noise spectrum proportional to (9) for some fixed exponent β. If that were indeed the case, our statistical efforts ought to be directed toward estimating the two-parameter set consisting of β and the constant of proportionality. Instead, as usual, we find ourselves using parametric tools to evaluate the confidence of a nonparametric statistic. The value of edf V depends on β. What can we do in the presence of a compound phase noise model

$$S_x^+(f) = \sum g_\beta \sin^\beta(2\pi f \tau_0), \tag{15}$$

a finite sum of fractional-difference spectra? Some help is given by the following theorem, which, although weak and perhaps obvious, is better than nothing.

Theorem. Let the phase noise be a finite sum of independent component noises with stationary Gaussian mean-zero second increments. Form an MVAR estimator V from the given phase noise, and corresponding estimators V_k from the components. Then

$$\text{edf } V \geq \min_k \text{ edf } V_k.$$

In other words, we never do worse than the worst component.

To apply this theorem to the situation (15), assume that the component β values are all in some subinterval of [-4,0] (the whole range, perhaps) . Use (14) and Table 2 to compute edf V_β for each "tabulated β in the subinterval, and take the smallest value as a conservative estimate of edf V. For example, if one believes that the noise has components between white phase and flicker phase, perhaps from prior knowledge, perhaps from a log-log σ-τ plot with slopes between -3/2 and -1, then one can minimize (14) over the first three rows of Table 2.

This theorem can be generalized to AVAR estimators and other situations involving averages of the square of a stationary Gaussian mean-zero process. Its usefulness for MVAR, as opposed to AVAR, is enhanced by the relatively weak dependence of MVAR estimator edf on β, as can be seen in Table 1. Similar tables for fully overlapped AVAR estimators [5][11] show a sharper dependence on β, especially for large τ/τ_0. Thus, minimizing estimator edf over β causes a smaller loss of accuracy for MVAR than for AVAR.

Concluding Remarks; Future Work

The previous paper on the third-difference approach [6] showed that MVAR estimates are almost as easy to calculate as AVAR estimates. The results given here extend this conclusion to the exact formulas for the confidence of the estimators. In addition, the approximation formulas for MVAR confidence are simpler and more uniform than existing approximation formulas for AVAR confidence [5][9], and the confidence values are more robust to spectral uncertainties. Having overcome the apparent increase in complexity of the extra moving-average filter in MVAR, we are free to enjoy all its advantages.

The problem of frequency drift removal now needs to be addressed. For AVAR, it is known that estimation of drift rate from the. data themselves, and removal therefrom, causes negative estimator bias that worsens as averaging time τ increases. The use of three-point [13] [14] or four-point [4] drift estimators, which extract a quadratic component of the time-residual sequence x_n, simplifies the calculation of the mean and variance of AVAR estimators with drift removed. I have no doubt that similar calculations for MVAR estimators can be made by using four-point drift estimators that extract a cubic component of the cumulative-sum sequence w_n.

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Table 1. Values and Approximation Errors for MVAR Estimator edf

N = 1024

m	r	1	M	P	β = 0.0	-1.0	-2.0	-3.0	-4.0	
1	1	1	1022	1022.	525.9	589.3	681.6	828.6	1022.	
					+0.0	+0.0	+0.0	+0.0	+0.0	%
2	1	2	510	510.0	262.6	310.1	380.8	459.1	432.3	
	2	1	1019	509.5	477.0	496.5	515.2	523.6	441.4	
					-0.1	-0.1	-0.1	-0.1	-0.1	%
3	1	3	339	339.0	174.6	210.3	260.1	304.4	271.0	
	3	1	1016	338.7	373.9	349.9	341.5	334.6	274.0	
					+11.1	-2.8	-3.9	-4.1	-5.0	%
16	1	16	62	62.00	32.15	39.57	48.69	55.29	47.55	
	2	8	123	61.50	58.06	59.26	59.68	58.73	47.60	
	4	4	245	61.25	72.74	61.99	59.93	58.57	47.43	
					+4.1	+0.1	-0.2	-0.3	-0.2	%
	8	2	489	61.13	77.60	62.26	59.84	58.46	47.33	
					-2.6	-0.6	-0.2	-0.3	-0.2	%
	16	1	977	61.06	78.88	62.26	59.78	58.40	47.29	
					-4.3	-0.7	-0.2	-0.3	-0.2	%
128	1	128	6	6.000	3.375	4.061	4.909	5.552	4.766	
	2	64	11	5.500	5.754	5.841	5.857	5.716	4.535	
	4	32	21	5.250	7.005	5.922	5.706	5.525	4.367	
					+3.4	+0.4	-0.1	-2.3	+0.2	%
	8	16	41	5.125	7.354	5.840	5.599	5.417	4.277	
					-3.6	-0.3	-0.3	-2.5	+0.0	%
	16	8	81	5.063	7.410	5.784	5.542	5.361	4.231	
					-5.3	-0.4	-0.4	-2.6	+0.0	%
	32	4	161	5.031	7.405	5.755	5.513	5.332	4.207	
					-5.8	-0.4	-0.4	-2.6	+0.0	%
	64	2	321	5.016	7.394	5.739	5.498	5.318	4.196	
					-5.9	-0.4	-0.4	-2.6	+0.0	%
	128	1	641	5.008	7.386	5.732	5.491	5.311	4.190	
					-5.9	-0.4	-0.4	-2.6	+0.0	%
N = 16										
1	1	1	14	14.00	7.475	8.327	9.561.	11.51	14.00	
					-3.7	-3.1	-2.4	-1.4	+0.0	%
2	2	1	11	5.500	5.754	5.946	6.117	6.146	5.061	
					-10.6	-10.0	-9.2	-8.1	-5.9	%
3	3	1	8	2.667	3.815	3.526	3.386	3.224	2.508	
					+9.9	-2.0	-3.0	-7.2	-3.5	%
			noise type:		wh ph	fl ph	wh fr	fl fr	rw fr	

'Table 2. "Coefficients for Approximating MVAR Estimator edf

noise type	β	m = 1: a.	m = 1: al	m = 2: a.	m = 2: al	m > 2: a.	m > 2: al
wh ph	0.0	.51429	0	.93506	0	1.2245	.58929
	-0".5	.54277		.95407		1.0739	.59605
fl ph	-1.0	.57640		.97339		1.0030	.60163
	-1.5	.61688		.99246		.97"732	.59769
wh fr	-2.0	.66667		1.0101		.96774	.57124
	-2.5	.72948		1.023-7		.96102	.50974
fl fr	-3.0	.81057		1.0266		.94663	.41643
	-3.5	.91389		.99981		.90604	.34276
rw fr	-4.0	1.0000		.86580		.76791	.41115