

A Derivation of the Dick Effect From Control-Loop Models for Periodically Interrogated Passive Frequency Standards*

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Abstract

The phase of a frequency standard that uses periodic interrogation and control of a local oscillator (LO) is degraded by a long-term random-walk component induced by downconversion of LO noise into the loop passband. The Dick formula for the noise level of this degradation can be derived from explicit solutions of two LO control-loop models. A summary of the derivations is given here.

1 Introduction

In 1987, following a suggestion of L. Cutler, G. J. Dick [1] described a source of long-term instability for a class of passive frequency standards that includes ion traps and atomic fountains. In these standards, the frequency of a local oscillator (LO) is controlled by a feedback loop whose detection

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and control operations are periodic with some period T_c . For each cycle, the output of the detector is a weighted average of the 10 frequency error over the cycle. The weighting function $g(t)$, derived from quantum-mechanical calculations, depends on the method by which the atoms are interrogated by the RF field generated by upconversion of the 10 signal to the atomic transition frequency [1, 2, 3]. In general, $g(t)$ can be zero over a considerable portion of the cycle. The 10 control signal over each cycle is a function of the detector outputs from previous cycles.

The purpose of a frequency-control loop is to attenuate the frequency fluctuations of the 10 inside the loop passband, while tolerating them outside the passband. As Dick saw, though, the periodic interrogation causes out-of-band LO noise power, near the cycle frequency $f_c = 1/T_c$ and its harmonics, to be downconverted into the loop passband, thus injecting random false information about the current average LO frequency into the control signal. This random false frequency correction causes a component of white FM, or random walk of phase, to persist in the output of the locked 10 (10) over the long term. Dick gave a formula for the white-FM noise level contributed by this effect, namely,

$$S_y^{10}(0) = 2 \sum_{k=1}^{\infty} \frac{g_k^2}{g_0^2} S_y^{10}(kf_c), \quad (1)$$

where $S_y^{10}(f)$ is the spectral density of the 10 Dick-effect portion of the normalized frequency noise of the 10, $S_y^{10}(f)$ is the spectral density of the normalized frequency noise of the free-running 10, and g_k is the Fourier coefficient

$$g_k = \frac{1}{T_c} \int_0^{T_c} g(t) \cos(2\pi k f_c t) dt, \quad (2)$$

where $g(t)$ is assumed to be symmetric about $T_c/2$. This level of white FM near Fourier frequency zero contributes an asymptotic component of Allan variance given by

$$\sigma_y^2(\tau) \sim \frac{S_y^{10}(0)}{2\tau} \quad (f_c\tau \gg \infty) \quad (3)$$

My purpose here is to supplement previous derivations [1, 2, 3, 4] of the 10 Dick formula (1) by an approach that uses explicit time-domain solutions of simple 10 control loop models with a general detection weighting function $g(t)$. Careful interpretation of these solutions yields formulas for the 10

spectral density, and conditions for the validity of the Dick formula. These models are not represented to be realistic models of actual frequency standards. By exhibiting the presence of the Dick effect in models of transparent simplicity, I intend to remove any remaining doubt of its existence and to isolate its essential nature, in the hope of aiding efforts to reduce it.

This paper gives only a summary of the solution method; details will be submitted elsewhere.

2 Control-Loop Models

Figure 1 shows two models for an LO control loop. Model 1 is intended to correspond to Dick's models [1, 2]. Model 2 extends the model of Lo Presti, Patanè, Rovera, and De Marchi [4] to a general weighting function. A unified treatment of the two models is presented at the expense of a conflict of notation between this paper and [4]: because the model of Lo Presti et al. includes the effect of alternate interrogation of the two sides of a Ramsey fringe, the cycle period T_c used here corresponds to the sample period $T_s = 2T_c$ in [4], and the $g(t)$ used in Model 2 really consists of two periods of the $g(t)$ used in Model 1.

In Model 1, the box G_1 represents a linear time-invariant filter with impulse response $g(t) / (T_c g_0)$ for $0 < t < T_c$, and zero elsewhere. It is important to observe that G_1 has unity gain at DC. Its transfer function is

$$G_1(f) = \frac{1}{T_c g_0} \int_0^{T_c} g(t) e^{-i2\pi ft} dt. \quad (4)$$

The output of the box G_1 at time t is

$$G_1 y_{\text{LLO}}(t) = \frac{1}{T_c g_0} \int_0^{T_c} g(u) y_{\text{LLO}}(t - u) du,$$

which is fictitious unless t is a multiple of T_c . The output of the sampler at time nT_c is the normalized interrogation report

$$G_1 y_{\text{LLO}}(nT_c) = \frac{1}{T_c g_0} \int_0^{T_c} g(u) y_{\text{LLO}}(nT_c - u) du. \quad (5)$$

for the n th cycle. (Recall the symmetry of $g(t)$ about the midpoint of the cycle.) The detection noise term v_n can represent photon-count fluctuations

in frequency standards with optical detection, for example. The cumulative sum of the error signals, multiplied by a gain factor λ between 0 and 1, is the frequency correction y_n that is applied to the LO during the next cycle. Except for initial conditions, Model 1 is specified completely by (5) and the equations

$$y_n - y_{n-1} = \lambda (G_1 y_{LO}(nT_c) - v_n), \quad (6)$$

$$y_{LLO}(t) = y_{LO}(t) + y_{n-1}, \quad (n-1)T_c < t \leq nT_c, \quad (7)$$

in which it is convenient to suppose that n runs through all integers.

In Model 2, the hold and integration emits a delayed linear interpolation of the cumulative sum of the input to the hold, modulo a constant of integration. Let y_n be λ times that cumulative sum. Then y_n again satisfies (6). In place of (7) we have

$$y_{LLO}(t) = y_{LO}(t) + \left(\frac{t}{T_s} - n + 1\right) y_{n-1} + \left(n - \frac{t}{T_s}\right) y_{n-2}, \quad (n-1)T_c < t \leq nT_c. \quad (8)$$

In Model 1, the frequency correction during a cycle is constant; here, it is a ramp.

3 Summary of Solution Method

The derivation of the LLO frequency spectrum from these model equations is carried out by the following steps.

First, by isolating the digital aspects of the models, one can solve for y_n . In Model 1, substitution of (7) into (6) gives a first-order difference equation for y_n in terms of the quantity

$$w_n = G_1 y_{LO}(nT_c) - v_n. \quad (9)$$

The solution of this difference equation has the form $y_n = H_{d1} w_n$, where H_{d1} is a unity-gain lowpass digital single pole filter with transfer function

$$H_{d1}(z) = \frac{\lambda}{1 - (1 - \lambda)z^{-1}}$$

The time constant is approximately T_c/λ for $\lambda \ll 1$. The transient component of the solution is neglected. Model 2 gives a second-order difference equation that is solved by the two pole filter

$$H_{d2}(z) = \frac{\lambda}{1 - \phi_1 z^{-1} - \phi_2 z^{-2}}$$

4 Main and Aliased Spectra

Consider the main part (10) of the 1,1,0 frequency spectrum. The 1,0 spectrum is multiplied by a factor that is $O(f^2)$ as $f \rightarrow 0$. This is the basic action of the first-order frequency control loop which attenuates the excursions of the 1,0 inside the loop bandwidth. For example, flicker FM in the 1,0 is reduced to flicker FM in the 1,1,0, and random walk FM is reduced to white FM. In addition, there is a low-pass-filtered white detection noise in the 1,1,0 frequency. We can regard $H_c(z)G_1(f)$ as the closed-loop transfer function from 1,0 frequency noise to 1,0 correction signal.

The Dick effect is supposed to come from a white-FM component in the aliased spectrum. There is such a contribution if the aliased spectrum (11) is continuous and positive at $f = 0$. Under reasonable conditions, this is so, and we may set $f = O(z^{-1})$ in (11). Because $H_c(1) = 1$, we have

$$S_y^1(0) = 2 \sum_{k=1}^{\infty} |G_1(kf_c)|^2 S_y^{1,0}(kf_c), \quad (12)$$

where we have now used the symmetry of the summands about zero frequency. This formula holds for one-sided spectral densities also.

The numbers $|G_1(kf_c)|^2$ are invariant to cyclic translations of the function $g(t)$ in time. It follows that the result (12) is invariant to shifts in the time origin, i.e., if the 1,1,0 phase is sampled on any time grid with spacing T_c , then the samples will include a white-FM component with spectral density (12) at zero frequency. If $g(t)$ is symmetric about $T_c/2$ for our time origin, then

$$G_1(kf_c) = \frac{g_k}{g_0},$$

where g_k is given by (2). Thus (12) extends the Dick formula (1) to asymmetric weighting functions.

The Dick formula, which gives the *limiting* value of spectral density at zero Fourier frequency, is exact for both models, even though the 1,1,0 spectrum at nonzero frequencies is different for the two models. A simple approximation for the aliased spectrum (11) holds if the gain constant λ is much less than 1. Then the loop bandwidth is much less than f_c (time constant much greater than T_c). Assume also that $G_1(kf_c - f)$ and $S_y^{1,0}(kf_c + f)$ can be regarded as approximately constant for nonzero k and for f within the loop bandwidth. Then, for such f , the aliased spectrum has approximately the same shape as the frequency response of the digital filter H_c , with value

at 0 given by the Dick formula. For both models, this shape is Lorentzian. Thus, the Dick-effect Allan variance component takes the asymptotic white-M form (3) only for averaging times τ greater than roughly twice the loop time constant. In this approximation, the Dick-effect and detection noises appear inside the loop bandwidth, the non-aliased LO noise outside.

5 Remarks

Although I have not considered any other models, the Dick effect appears to be an inherent property of periodic local-oscillator control loops. For the two models treated here, this was shown by a careful interpretation of explicit solutions for the output frequency as function of time.

I have now come full circle on this topic. My involvement began in 1987 when John Dick asked me to derive the spectrum of $G_1 y_{LO}$ after sampling. I did not understand: after all, G_1 is applied to y_{LLO} , not y_{LO} . Nevertheless, I did the calculation, thereby contributing the factor 2 in Dick's formula. Now, in Fig. 2, we see how the sampled $G_1 y_{LO}$ fits into the picture. Could the Dick effect be cancelled by replacing the averaging filter in the upper branch of Fig. 2 by a G_1 filter? Alas—the existence of the block diagram in Fig. 2 is mathematical, not physical.

References

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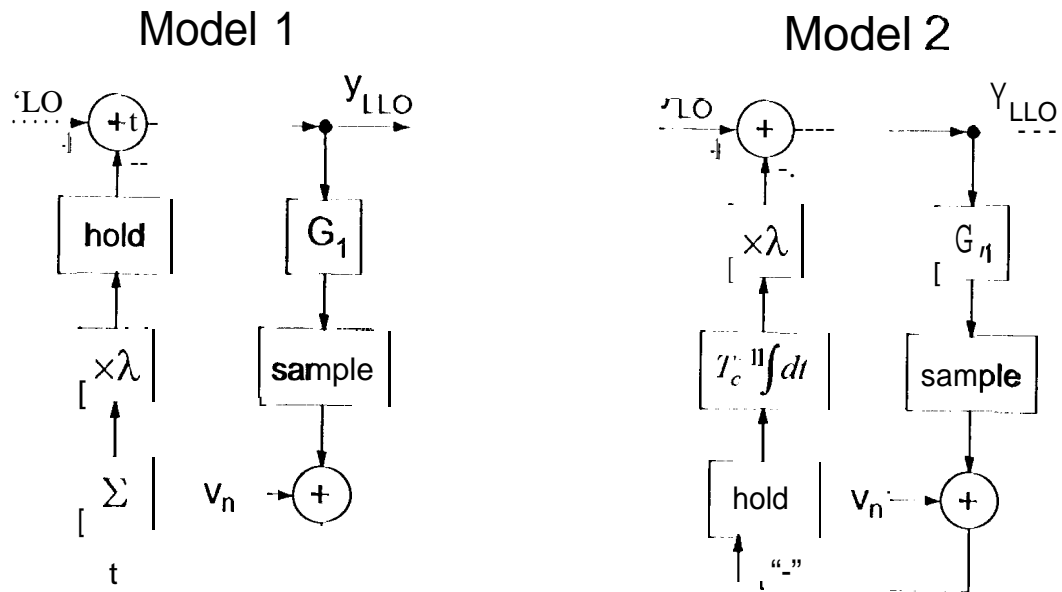
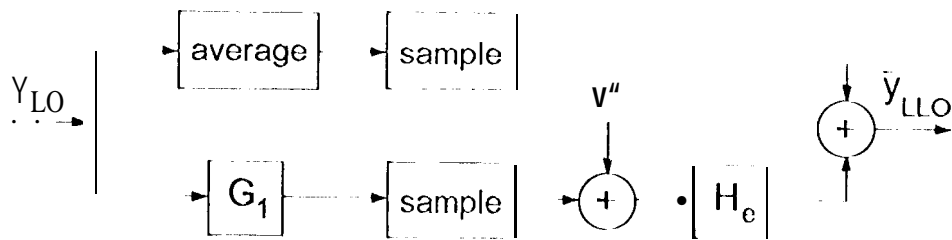


Fig. 1. Simplified models of local-oscillator control loops



$$\text{Model 1: } H_e(z) = \frac{\lambda z^{-1}}{1 - (1 - \lambda) z^{-1}}$$

$$\text{Model 2: } H_e(z) = \frac{\lambda(z^{-1} + z^{-2})}{1 - \phi_1 z^{-1} - \phi_2 z^{-2}}$$

Fig. 2. Solution of both models for 1 L.O frequency averages