AN LTI/LTV DECOMPOSITION OF ADAPTIVE FEEDFORWARD SYSTEMS WITH SINUSOIDAL REGRESSORS

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Abstract

It is shown that a large class of adaptive feedforward controllers having a sinusoidal regressor (i.e., a regressor comprised exclusively of sinusoidal signals), can be written as the parallel connection of a purely linear time-invariant (LTI) subsystem and a linear time-varying (LTV) subsystem. Under certain stated conditions the LTV subsystem vanishes and the system is purely LTI. In this case, the adaptive control analysis and design can be performed completely using LTI methods. In the case where the LTV subsystem does not completely vanish, an explicit upper bound is established on the induced 2-norm of the LTV block which allows systematic analysis using robust control methods. The upper bound can be maximally tightened by solving a convex optimization problem involving a related linear matrix inequality (LMI). This overall approach represents a strong departure from using the standard Lyapunov and Hyperstability methods for adaptive control analysis, and provides a more complete robustness analysis to ensure that this class of adaptive systems can be confidently implemented in practice,

1 INTRODUCTION

A large number of adaptive systems used in practice (e.g., for adaptive signal processing, noise canceling) acoustics, vibration suppression, etc.), have regressors which contain sinusoidal excitations. In certain cases, such systems have been found to admit exact finite- dime~]siona.l linear time-invariant (LTI) representations [7] [1 O] [1 1][5] [18]. Recently, a necessary and sufficient condition, denoted as the XO condition [I] [2], has been found which characterizes precisely when such LTI representations exist. Such cases are important because in contrast to nonlinear and/or time-varying representations, the stability, convergence and robustness properties of LTI systems can be completely characterized using standard methods.

Unfortunately, it is not always possible to satisfy the XO condition exactly. For example, it has been shown in Bayard [1] that a regressor formed by filtering sinusoids through a Tap Delay Line (TDL) will only satisfy the XO condition exactly in the limit as the number of taps increases to infinity. As another example, implementations based on paired sine/cosine regressors will satisfy the XO condition exactly only if successive pairs of elements of the regressor are perfectly matched with respect to gain and are exact 1 y 90 degrees out of phase. Clearly, in practice it is difficult to satisfy the XO condition exactly, Hence, it is desired to establish representations and robustness analysis for the general case.

In the present paper, it is shown that a large class of adaptive systems with sinusoidal regressors can be written as the parallel connection of a purely LTI block and an LTV Mock. An explicit upper bound is given on the induced 2-norm of the LTV block which allows systematic analysis using robust control methods. The upper bound can be maximally tightened by solving a related convex optimization problem involving a linear matrix inequality (LM 1), and solved using available software packages. As desired (for consistency with earlier results), the norm-bound vanishes when the XO condition is satisfied leading to a purely LTI system, An example is given showing a complete robustness aria.lysis of an adaptive system with a single-tone sine/cosine regressor where there are errors in the implementation. All results in this paper are based on the anal ysis in a recent report [1].

2 BACKGROUND

2.1 Adaptive Systems with Harmonic Regressors

The configuration to be studied is shown in Figure 2.1. An estimate $\hat{y}$ of some signal y is to be constructed as a linear combination of the elements of a regressor vector $x(t) \in$ RN, i.e.,

Estimated Signal

$$\hat{y} = w(t)^T x(t) \tag{2.1}$$

where w(t) $\in R^N$ is a parameter vector which is tuned in real-time using the adaptation algorithm,

Adaptation Algorithm

$$w = \mu\Gamma(p)[\tilde{x}(t)e(t)] \tag{2.2}$$

Here, the notation $\Gamma(p)[\cdot]$ is used to denote the multivariable LTI transfer function I'(s) $\cdot I$ where I'(s) is any LTI transfer function in the Laplace s operator (the differential operator p will replace the Laplace operator s in all time-domain filtering expressions); the term $e(t) \in R^1$ is an error signal; $\mu > 0$ is an adaptation gain; and the signal $\tilde{x}$ is obtained by filtering the regressor x through any stable filter $'(p), i.e.,

Regressor Filtering

$$\tilde{x} = F(p)[x] \tag{2.3}$$

The notation $F(p)[\cdot]$ denotes the multivariable LT1 transfer function $F(s) \cdot I$ with SISO filter $F(s)$, acting on the indicated vector time domain signal.

For the. purposes of this paper, it will be assumed that the regressor x can be written as a linear combination of m distinct sinusoidal components $\{\omega_i\}_{i=1}^m$, O < $\omega_1 < \omega_2 < \ldots < \omega_m$, where the frequencies have been ordered by size from smallest to largest. Equivalentl y, it is assumed that there exists a matrix $\mathcal{X} \in R^{N \times 2m}$ such that,

Harmonic Regressor

$$x = \mathcal{X}c(t) \tag{2,4}$$

$$c(t) = [\sin(\omega_1 t), \cos(\omega_1 t), ..., \sin(\omega_m t), \cos(\omega_m t)]^{T} \in R^{2m} \tag{2.5}$$

Equations (2. 1)-(2.5) taken together will be referred to as a *harmonic adaptive system*. Collectively, these equations define an important open-loop mapping from the error signal c to the estimated output $\hat{y}$. Because of its importance, this mapping will be demoted by the special character $\mathcal{H}$, i.e.,

$$\hat{y} = \mathcal{H}[e] \tag{2.6}$$

The special structure of $\mathcal{H}$ is depicted in Figure 2.1.

Most generally $\mathcal{H}$ is a linear time-varying (LTV) operator. However, the main results of this paper show that $\mathcal{H}$ can always be written as the parallel connection of an LT1 subsystem and an LTV subsystem where the "size" of the LTV subsystem is ch aracterized in terms of a norm bound. This provides an important alternative representation of the adaptive system, particularly in cases (often occuring in practice) where the norm bound on the LTV perturb ation is small.

REMARK 2.1 The definition of $\Gamma(s)$ is left intentionally general to include analysis of the gradient algorithm (i.e., with the choice I'(s) = 1/s), the gradient algorithm with leakage (i.e., I'(s) $= 1/(s+\sigma); \sigma \geq$ O), proportional-plus-integral adaptation (i.e., I'(s) = $k_p + k_i/s$), or arbitrary linear adaptation algorithms of the designer's choosing, Adaptation laws which

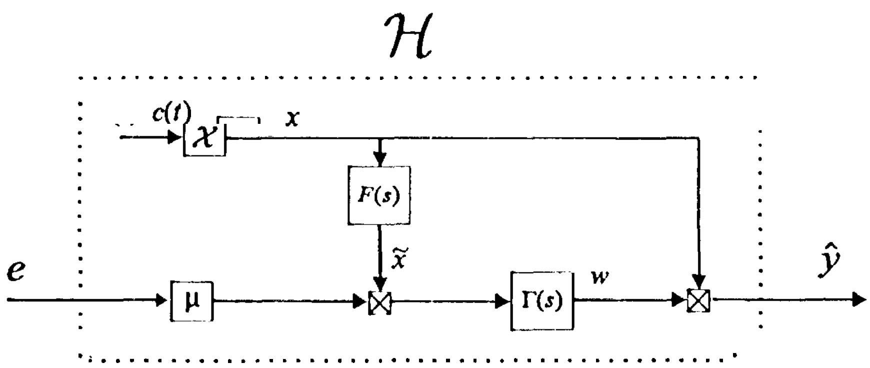

Figure 2.1: LTV operator $\hat{y} = \mathcal{H}[e]$ for adaptive system with harmonic. regressor x, adaptation law $\Gamma(s)$, and regressor filter $F(s)$.

are nonlinear or normalized (e.g., divided by the norm of the regressor), are not considered here since they do not have an equivalent LTI representation I'(s). ●

REMARK 2.2 The use of the regressor filter $F(s)$ is (2.3) allows the unified treatment of many important adaptation algorithms including the well-known Filtered-X algorithm from the signal processing literature [16] [11] [4] [1 7], and the Augmented Error algorithm of Monopoli [9]. Since x is comprised purely of sinusoidal components and F in (2.3) is stable, all subsequent analysis will assume that the filter output $\tilde{x}$ has reached a steady-state condition, ■

3 LT1 REPRESENTATIONS

The following result taken from [I] [2] will be needed, which gives necessary and sufficient conditions for the operator $\mathcal{H}$ to be LTI.

THEOREM 3.1 (LTI Representation Theorem) *Let the regressor $x(t)$ in the adaptive system (21) -(2. S) be given by the general multitone harmonic expression (2.4)(2.5) where the frequencies $\{\omega_i\}_{i=1}^m$ are distinct, nonzero, and $|F(j\omega_i)| > 0$ for all i.*

Then,

(i) *The mapping $\mathcal{H}$ from e to $\hat{y}$ is exactly representable as the linear time-invariant operator,*

$$\mathcal{H}: \quad \hat{y} = \bar{H}(p)e \tag{3.1}$$

if and only if the matrix $\mathcal{X}$ in (2.4) satisfies the *following X- Orthogonality (XO) condition,*

$\mathcal{X}$- Orthogonality (XO) Condition:

$$\mathcal{X}^T\mathcal{X} = D^2 \tag{3.2}$$

$$D^2 \triangleq \begin{bmatrix} d_1^2 \cdot I_{2\times2} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_m^2 \cdot I_{2\times2} \end{bmatrix} \in R^{2m\times 2m} \tag{3.3}$$

where, $d_i^2 \geq 0, i = 1, \ldots m$ are scalars and $I_{2\times2} \in R^{2\times2}$ is the matrix identity.

(ii) H(s) in *(9.1) is* given *in closed-form as,*

$$\overline{H}(s) = \mu \sum_{i=1}^{m} d_i^2 \cdot H_i(s) \tag{3.4}$$

$$H_i(s) = \frac{F_R(i)}{2}\Big(\Gamma(s - j\omega_i) + \Gamma(s + j\omega_i)\Big) + \frac{F_I(i)}{2j}\Big(\Gamma(s - j\omega_i) - \Gamma(s + j\omega_i)\Big) \tag{3.5}$$

$$F_R(i) \triangleq \mathrm{Re}(F(j\omega_i)); \; F_I(i) \triangleq \mathrm{Im}(F(j\omega_i)) \tag{3.6}$$

DEFINITION 3,1 *The matrix $\mathcal{X}^T\mathcal{X} = D^2$ having the special pairwise diagonal structure (S'. 9) in Theorem S'.1 is defined as the* **confluence matrix** *associated with a particular harmonic adaptive system (2.* 1)-(2.5), ∎

Without loss of generality, the confluence matrix will be assumed to be nonsingular, i.e., $D^2 > 0$, since any zero diagonal pair $d_i^2 \cdot I_{2\times2}$, $d_i^2 = 0$ in $D^2 > 0$ corresponds to a distinct frequency ω_i which can be removed from the definition of *c(t) in* (2.5), reducing the value of m accordingly,

DEFINITION 3.2 A *minimal* realization of an *LTI harmonic adoptive system (2,1)-(2.5) is defined by the regressor choice $x_1 = \mathcal{X}_1 c(t)$ where $\mathcal{X}_1 \in R^{2m\times2m}$ is any square matrix factor of its confluence matrix $D^2 > 0$, i. e.,*

∎

DEFINITION 3.3 Tonal canonical form *is defined as the unique minimal realization of an LTI harmonic adaptive system* (2.1)-(2.5) *specified by the regressor choice $x_1 = \mathcal{X}_1 c(t)$ where $\mathcal{X}_1 \in R^{2m\times 2m}$ is the unique positive diagonal square-root $\mathcal{X}_1 = D > 0$ of its confluence matrix D^2.* ∎

The following Corollary to Theorem 3.1 will be useful for a later example,

COROLLARY 3.1 (Gradient Algorithm with Leakage) *Assume that the adaptive system with harmonic regressor (2. 1)-(2. 5) is specified as the gradient adaptive algorithm with leakage, a. e.,*

$$\dot{w} = -\sigma w + x(t)e(t) \tag{3.7}$$

for some value *of the leakage* parameter $\sigma \geq 0$ *(cf., Ioannou and Kokotovic [8]). Then, if the XO condition of Theorem S.1 is satisfied, the LTI expression (3.4) for 'H is given by,*

$$\bar{H}(s) = \mu \sum_{i=1}^{m} d_i^2 \frac{s+\sigma}{s^2 + 2\sigma s + (\omega_i^2 + \sigma^2)} \tag{3.8}$$

PROOF: Result (3.8) follows by substituting, I'(s) $= \frac{1}{s+\sigma}$; $\sigma \geq$ O, and F(s) = 1 in Theorem 3.1, and rearranging.

∎

4 THE LTI/LTV DECOMPOSITION

The main result is given next, showing that in the general case where the XO condition is not satisfied, the mapping $\mathcal{H}$ can always be decomposed into a *parallel connection* of an LTI subsystem and an LTV perturbation,

THEOREM 4.1 (LTI/LTV Decomposition) Consider *-the adaptive system (2.1)-(2.9) with harmonic regressor (2.4)(2. 5). Then,*

(i) *In general the mapping $\mathcal{H}$ from e to $\hat{y}$ can be expressed as the parallel connection of an LTI block $\bar{H}(s)$, and an LTV perturbation block* $\tilde{\Delta}$,

$$\mathcal{H}: \quad \hat{y} = \bar{H}(p)e + \tilde{\Delta}[e] \tag{4.1}$$

where,

$$\bar{H}(s) \triangleq \mu \sum_{i=1}^{m} d_i^2 \cdot H_i(s) \tag{4.2}$$

$$\tilde{\Delta}[e] \triangleq \mu e(t)^T \Delta \Gamma(p) \left[\mathcal{F} e(t) e \right] \tag{4.3}$$

$$\Delta \triangleq \mathcal{X}^T \mathcal{X} - D^2 \tag{4.4}$$

$$\mathcal{F} \triangleq blockdiag\{\mathcal{F}_i\} \in R^{2m \times 2m} \tag{4.5}$$

$$\mathcal{F}_i \triangleq \begin{bmatrix} F_R(i) & F_I(i) \\ -F_I(i) & F_R(i) \end{bmatrix} \in R^{2\times 2}; \qquad \textit{for } i = 1, \ldots, m \tag{4.6}$$

$$F_R(i) \triangleq \mathbf{Re}(F(j\omega_i)); F_I(i) \triangleq \mathbf{Im}(F(j\omega_i)) \tag{4.7}$$

and where $H_i(s)$ is as defined in (3.5) of Theorem 3.1, and D^2 is chosen (non-uniquely) as any matrix of the 2 × 2 block-diagonal form (3.3).

(ii) If the adaptation law $\Gamma(s)$ is stable with infinity norm $||\Gamma(s)||_\infty$, then the gain of the LTV perturbation can be bounded from above as,

$$||\tilde{\Delta}||_{2i} \leq \mu m \bar{\sigma}(\Delta) ||\Gamma(s)||_\infty \max_i |F(\omega_i)| \tag{4.8}$$

where $||\cdot||_{2i}$ denotes the induced $\mathcal{L}_2$-norm of the indicated operator.

PROOF:

Proof of (i): Substituting (2.4) and the relation $\tilde{x} = F(p)[\mathcal{X}'c(t)] = \mathcal{X}\mathcal{F}c(t)$ into (2.1)-(2.3) gives,

$$\hat{y} = \mu x(t)^T \Gamma(p) \left[\tilde{x}(t) c\right] \tag{4.9}$$

$$= \mu c(t)^T \mathcal{X}'^T \mathcal{X}' \cdot \Gamma(p) \left[\mathcal{F} c(t) c\right] \tag{4.10}$$

Decompose $\mathcal{X}'^T \mathcal{X}$ into two distinct parts using the identity,

$$\mathcal{X}'^T \mathcal{X} = D^2 + (\mathcal{X}'^T \mathcal{X}' - D^2) = D^2 + \Delta \tag{4.11}$$

Substituting identity (4.11) into (4.10), and expanding gives two distinct subsystems,

$$\hat{y} = \mu c(t)^T D^2 \cdot \Gamma(p) \left[\mathcal{F} c(t) c\right] + \mu c(t)^T \Delta \cdot \Gamma(p) \left[\mathcal{F} c(t) c\right] \tag{4.12}$$

By the results of Theorem 3.1 the LTI part $H(s)$ is uniquely associated with the operator containing the D^2 term, and the LTV part $\tilde{\Delta}$ is uniquely associated with the operator containing the Δ term in (4.12).

Proof of (ii): This result follows by standard signal norm bounding methods, and only a brief outline is given. Let,

$$y_\Delta \triangleq \tilde{\Delta}[c] = \mu c^T \Delta \Gamma(p)[\mathcal{F} c e] = \mu ||\beta^T \eta||_2 \tag{4.13}$$

where,

$$\beta \triangleq \Delta^T c \tag{4.14}$$

$$\eta \triangleq \Gamma(p)[\mathcal{F} c e] \tag{4.15}$$

and the 2-norm is defined as $||x||_2 \triangleq \left[\int_0^\infty x^T x dt\right]^{\frac{1}{2}}$. Then,

$$||y_\Delta||_2 = \mu||\beta^T\eta||_2 \leq \mu||\eta||_2 \max_t(\beta^T\beta)^{\frac{1}{2}} \tag{4.16}$$

But it can be shown that,

$$\max_t(\beta^T\beta)^{\frac{1}{2}} \leq m^{\frac{1}{2}} \,.\, \bar{\sigma}(\Delta) \tag{4.17}$$

and,

$$||\eta||_2 \leq ||\Gamma(s)||_\infty m^{\frac{1}{2}} \max_i |F(\omega_i)| \cdot ||e||_2 \tag{4.18}$$

Combining (4.16)(4,17) and (4.18) gives,

$$||y_\Delta||_2 \leq \mu m\bar{\sigma}(\Delta)||\Gamma(s)||_\infty \max_i |F(\omega_i)| \cdot ||e||_2 \tag{4.19}$$

Hence,

$$||\tilde{\Delta}||_{2i} \triangleq \sup_{e\in\mathcal{L}_2} \frac{||y_\Delta||_2}{||e||_2} \leq \mu m\bar{\sigma}(\Delta)||\Gamma(s)||_\infty \max_i |F(\omega_i)| \tag{4.20}$$

which is the desired result, ■

The LTI/LTV decomposition of $\mathcal{H}$ can be understood by the sequence of block diagram rearrangements shown in Figure 4.1. Specifically, Figure 4.1 Part a. shows the initial adaptive system with harmonic regressor; Part b. shows the matrix $\mathcal{X}$ pushed through several scalar matrix blocks of the diagram; Part c, uses the identity $\mathcal{X}^T\mathcal{X} = D^2 + (\mathcal{X}^T\mathcal{X} - D^2)$ to split the diagram into two subsystems; and Part d. recognizes the upper subsystem as LTI and the lower subsystem as LTV (from Theorem 3,1) with the indicated norm bound,

REMARK 4.1 The LTI/LTV decomposition in Theorem 4.1 is important for adaptive systems which do not exactly satisfy the XO condition. In this case, the adaptive system can be analyzed using modern robust control methods (i.e., small gain theorem) making use of the analytic expression (4.2) for the LT1 block II(s) and the norm bound (4,8) on the time-varying perturbation block A [15] [20], The induced $\mathcal{L}_2$-norm has been bounded here since it is consistent with the use of H_∞ theory for robustness analysis, However, other induced norms (e. g., $\mathcal{L}_p$-norm, $p = 1$ or $3 \leq p \leq \infty$) *can* be similarly bounded and may be of interest for other types of analysis [19], ■

REMARK 4.2 The need for $||\Gamma(s)||_\infty$ to exist in Theorem 4.1 (part ii) requires that the adaptive law uses some type of '{leakage" (cf., Ioannou and Kokotovic [8]). This condition is somewhat conservative in light of the fact that many adaptive laws without leakage are known to be stable in closed-loop from separate Lyapunov stability arguments [12], Hence, it is conjectured that a less conservative norm-bound is possible which does not

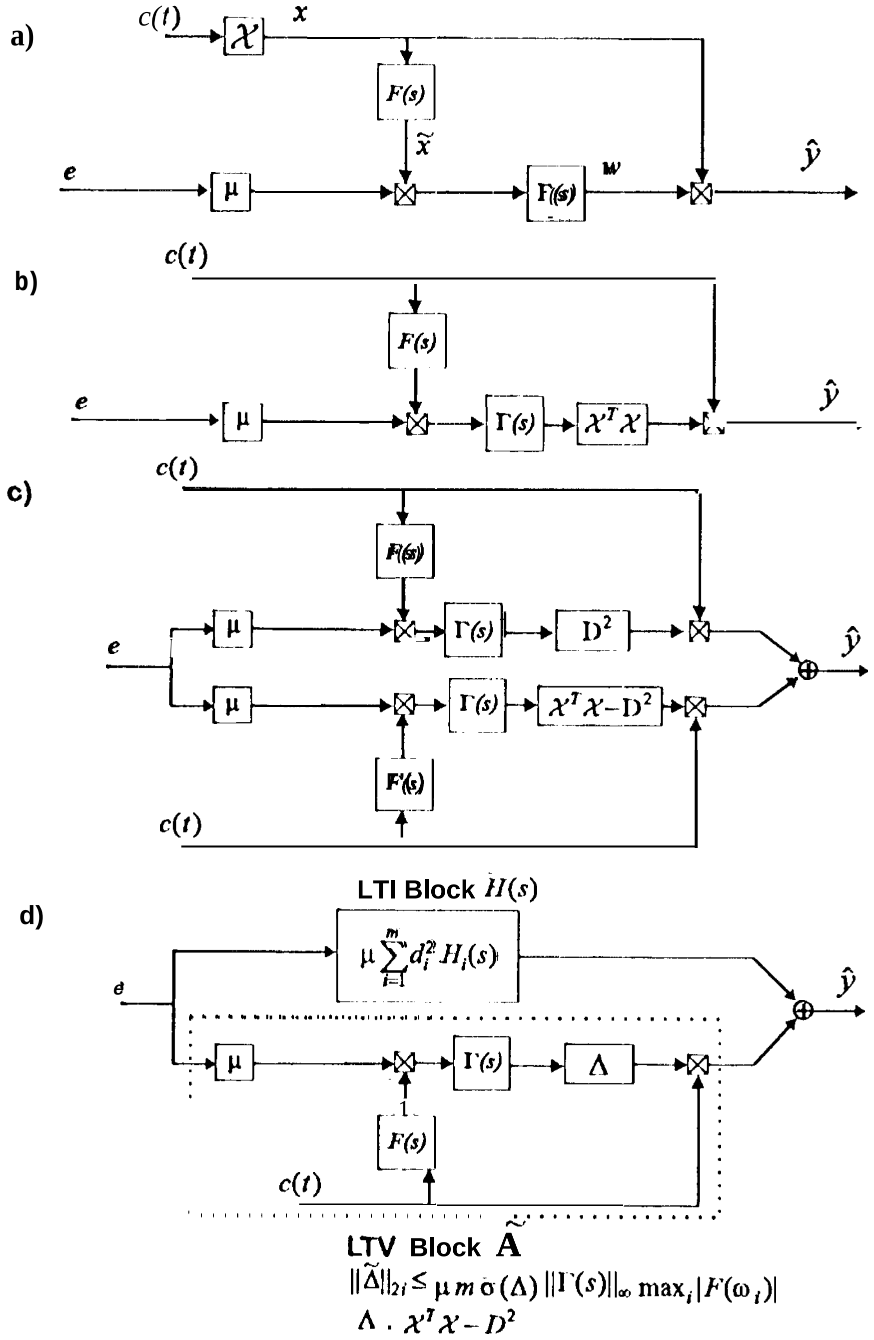

Figure 4,1: General LTI/LTV Decomposition of $\mathcal{H}$

require leakage. The search for such a bound (if it exists) is left as an open issue, However Theorem 4.1 will be applicable to most practical implementations since leakage is often added to ensure robustness. ■

5 OPTIMIZED NORM BOUNDS

The decomposition as stated in Theorem 4.1 is only unique for a specified choice of D^2. Hence, D^2 plays the role of a "multiplier" which should be optimized to capture "most" of the LTI character of the $\mathcal{H}$ operator in the LTI/LTV decomposition. The optimization problem will be addressed in this section,

The approach is to minimize the norm-bound (4.8) of the LTV operator over all possible D^2 of the appropriate pairwise diagonal form (3.3). Since the matrix D^2 only appears in the $\bar{\sigma}(\Delta)$ term, this is equivalent to minimizing $\bar{\sigma}(\mathcal{X}^T\mathcal{X} - D^2)$. The problem is stated below and shown to lead to a convex linear matrix inequality (LMI) optimization problem.

LEMMA 5.1 (LTV Norm-Bound Optimization) *Consider the following optimization problem,*

$$\min_{\mathcal{D}} \bar{\sigma}\left(\mathcal{X}^T\mathcal{X} - \mathcal{D}\right) \tag{5.1}$$

subject to,

$$\mathcal{D} \triangleq D^2 \triangleq \begin{bmatrix} d_1^2 \cdot I_{2\times 2} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & d_m^2 \cdot I_{2\times 2} \end{bmatrix} \in R^{2m\times 2m} \tag{5.2}$$

where, $d_i^2 \geq 0$, $i = 1, \dots, m$ are arbitrary scalars.

Then the solution is given by solving the following equivalent convex, optimization problem,

$$\min_{t,\mathcal{D}} \ t \tag{5.3}$$

subject to,

$$\begin{bmatrix} t \cdot I & \mathcal{X}^T\mathcal{X} - \mathcal{D} \\ \mathcal{X}^T\mathcal{X} - \mathcal{D} & t \cdot I \end{bmatrix} \geq 0 \tag{5.4}$$

$$t > 0 \tag{5.5}$$

$$\mathcal{D} \geq 0 \tag{5.6}$$

where $\mathcal{D}$ is constrained to have the pairwise diagonal structure (5.2)

PROOF: Consider the related optimization problem,

$$\min_{t} \ t \tag{5.7}$$

subject to,

$$\begin{bmatrix} t \cdot I & \mathcal{X}^T\mathcal{X} - \mathcal{D} \\ \mathcal{X}^T\mathcal{X} - \mathcal{D} & t \cdot I \end{bmatrix} \geq 0 \tag{5.8}$$

$$t > 0 \tag{5.9}$$

Given $t > 0$, inequality *(5.8) is* known to be equivalent to $S \geq 0$ where S is the Schur complement $t - (\mathcal{X}^T\mathcal{X} - \mathcal{D})^T(t^{-1} \cdot I)(\mathcal{X}^T\mathcal{X} - \mathcal{D})$ (cf., [3]), But inequality $S \geq 0$ is equivalent to the inequality $t^2 \geq (\mathcal{X}^T\mathcal{X} - \mathcal{D})^T(\mathcal{X}^T\mathcal{X} - D)$, which is minimized by $t = \bar{\sigma}(\mathcal{X}^T\mathcal{X} - \mathcal{D})$. The result of the lemma follows by further optimizing this solution over $\mathcal{D}$ with the constraint $\mathcal{D} \geq 0$.

■

The optimization problem (5.3)-(5,6) is in a standard form of a linear objective function with LMI constraints. As such, it can be solved using many available software packages for LMI problems, such as the LMI Control Toolbox [6] which implements the Projective Algorithm of Nesterov and Nemirovskii [13][1 4].

For single-tone problems, the optimal $D^2 = d^2 .1$ can be found analytically.

LEMMA 5.2 (Single-Tone Case) *Consider the optimization problem,*

$$\min_{d^2} \bar{\sigma}(\mathcal{X}^T\mathcal{X} - d^2 \cdot I_{2\times 2}) \tag{5.10}$$

where $d^2 \geq 0$ is an arbitrary scalar.

Then the solution d^2 is given by the average of the diagonals of $\mathcal{X}^T\mathcal{X}$, i.e.,

$$d^2 \triangleq \frac{1}{2}(m^{11} + m^{22}) \tag{5.11}$$

wh ere,

$$\mathcal{X}^T\mathcal{X} \triangleq \mathcal{M} \tag{5.12}$$

$$\mathcal{M} \triangleq \begin{bmatrix} m^{11} & m^{12} \\ m^{21} & m^{22} \end{bmatrix} \in R^{2\times 2} \tag{5.13}$$

PROOF: The singular values σ_1, σ_2 of the 2 x 2 symmetric matrix $A = \mathcal{X}^T\mathcal{X} - d^2, I$ can be written in terms of its eigenvalues as,

$$\sigma_i(\Delta) = |\lambda_i(\Delta)| = |\alpha_i(\mathcal{X}^T\mathcal{X}) - d^2|, \; i = 1,2 \tag{5.14.}$$

where $\lambda_i(\Delta)$ denotes an eigenvalue of A and $\alpha_i(\mathcal{X}^T\mathcal{X})$ denotes an eigenvalue of $\mathcal{X}^T\mathcal{X}$. Here, the eigenvalues λ_i and α_i are related by the shift in the complex plane i.e., $\lambda_i = \alpha_i - d^2$, $i = 1, 2$. Hence as d^2 is increased, the λ_i are determined by shifting the (nonegative real) eigenvalues α_i to the left along the real axis a distance of d^2, The quantity $\bar{\sigma}(\Delta) = max(|\lambda_1|, 1\sim_2 1)$ is clearly minimized at the point where $\lambda_1 = -\lambda_2$, or equivalent] y where,

$$d^2 = (\alpha_1 + \alpha_2)/2 = Trace(\mathcal{X}^T\mathcal{X})/2 \tag{5.15}$$

which is the desired result (5.11).

6 EXAMPLE

Consider the gradient adaptive algorithm with leakage, i.e.,

$$\dot{w} = -\sigma w + x(t)e(t) \tag{6.1}$$

for some value of the leakage parameter $\sigma \geq 0$. This corresponds to the choice $\Gamma(s) = 1/(s + \sigma)$, $F(s) = 1$ in the adaptive system (2.1)-(2.5).

The ideal sine/cosine regressor is defined by,

$$x = \begin{bmatrix} \sin \omega_1 t \\ \cos \omega_1 t \end{bmatrix}$$

Since $x = \mathcal{X}_1 c(t)$ with $\mathcal{X}_1 = diag[1, 1]$ it follows that the XO condition $\mathcal{X}_1^T \mathcal{X}_1 = D^2$ is satisfied exactly with confluence matrix $D^2 = diag[1, 1]$. Using the results from Corollary

Hence, by the LTI/LTV decomposition the adaptive system is representable by a parallel connection of the LTI Mock $\tilde{H}(s)$ given in (6.2) and an LTV perturbation block A with induced 2-norm bound,

$$\begin{aligned} ||\tilde{\Delta}||_{2i} &\leq \mu m \bar{\sigma}(\Delta)||\Gamma(s)||_\infty |F(j\omega_1)| \\ &= \frac{\mu}{\sigma}\left(\frac{\epsilon^2}{4}\cos^2(\theta) + \sin^2(\theta)\right)^{\frac{1}{2}} \end{aligned}$$

It is seen that as $\epsilon \to 0$ and $\theta \to 0$, the norm bound A goes to zero, which ensures a pure LTI representation in the limiting case of a perfect regressor implementation. For finite ϵ and θ the, above LTI/LTV decomposition is amenable to analysis using standard robust control methods.

7 CONCLUSIONS

This paper provides an alternative representation of adaptive feedforward systems with sinusoidal regressors. Specifically, the LTI/LTV Decomposition Theorem is proved which decomposes the adaptive system into a parallel connection of an LT1 subsystem and an LTV subsystem. An explicit norm-bound is established on the LTV subsystem, to enable the use of robust control methods applicable to LT1 systems with norm-bounded perturbations.

The LTI/LTV decomposition is unique up to the choice of a certain multiplier matrix D^2. This multiplier is ideally chosen to minimize the norm-bound on the LTV operator, It was shown that the optimal multiplier could be found by solving a related convex programming problem involving a linear matrix inequality, The LM1 problem is readily solved using available software. For the single-tone case, an analytic solution was provided.

A simple example was given to demonstrate the main ideas, using a regressor constructed from a non-ideal sine/cosine basis. The system was optimally decomposed into LTI and LTV subsystems, indicating that the LTV perturbation is on the order of phase and amplitude perturbations in the regressor. As desired, a perfect LT1 representation is recovered in the limit as the phase and gain distortions became small,

The LTI/LTV representation is significantly different from other representations of adaptive systems which typically require the usc of Lyapunov and Hyperstability methods for adaptive control analysis [12]. In contrast, the LTI/LTV representation can be analyzed and designed using modern robust control tools applicable to LTI systems with norm-bounded perturbations, It is hoped that this will open up a new understanding of how such adaptive systems work and speed up their reduction to practice.

8 ACKNOWLEDGEMENTS

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