Spectral Ambiguity of Allan Variance

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Abstract

We study the extent to which knowledge of Allan variance and other finite-difference variances determines the spectrum of a random process. The variance of first differences is known to determine the spectrum. We show that, in general, the Allan variance does not. A complete description of the ambiguity is given.

Real-valued random processes $x(t)$ with stationary nth differences

$$\Delta_\tau^n x(t) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j x(t - j\tau) \qquad (1)$$

are often used as models for observed functions of time t that grow in a nonstationary manner, but nevertheless seem to be driven by an underlying stationary stochastic mechanism, in that the character of their growth and local fluctuations does not change with time. The nth difference variance $D_n(\tau)$ is defined as the variance of the random variable $\Delta_\tau^n x(t)$, which, by the stationarity assumption, depends only on τ. The Allan variance is defined by $\sigma_y^2(\tau) = D_2(\tau)/(2\tau^2)$. It is often written as

$$\sigma_y^2(\tau) = \frac{1}{2\tau^2} \operatorname{var}\left[\int_{t-\tau}^{t} y(u)du - \int_{t-2\tau}^{t-\tau} y(u)du\right],$$

where $y = \dot{x}$.

Finite-difference variances have been used in several areas of science and engineering as statistical descriptions of time series. Kolmogorov (see Ref. [1], pp. 86–93) used $D_1(\tau)$ in studies of fluid turbulence. Barnes [2], Lindsey, and Chie [3] proposed the use of $D_n(\tau)$ for time deviation $x(t)$ to describe noise in frequency sources. For the same purpose, Barnes [2] and Allan [4] proposed an N-sample variance that became known as Allan variance, and (with $N = 2$) was adopted as the standard specification of frequency instability. Allan variance has also been used to detect $1/f$ noise in lasers [5], fractal exponents in spiking laser oscillation [6], $1/f$ noise in α and β decay [7] [8], and $1/f^2$ noise in quantum transport [9].

The spectrum of the process $x(t)$ maps to $\sigma_y^2(\tau)$ by a simple formula (see (2) below), by which a power-law spectral density $S(f) \propto f^\beta$ ($\beta < -1$) maps to $\sigma_y^2(\tau) \propto \tau^{-1-\beta}$. In applications, one often wishes to invert this mapping. The usual practice is to infer a spectral density $S(f)$ of form $\sum c_\beta f^\beta$ for $x(t)$ or $y(t)$ by observing linear regions in the log-log plot of $\sigma_y^2(\tau)$ vs. τ. There is no problem with this practice if the actual spectrum has this parametric form. On the other hand, *general* inversion formulas for $S(f)$ (or its Fourier transform) from $D_n(\tau)$ have also been given [3]. These formulas suggest that $D_n(\tau)$ *unambiguously* determines the spectrum. For D_1, this assertion is known to be true [1]; a proof is given below (see Theorem 1(a)). For D_2, the present study shows that the assertion is false in general (but true if D_1 also exists—see Theorem 1(b)). Figure 1 shows two examples of distinct spectra with the same D_2. For D_n with $n > 2$, the problem is open, as far as the author knows.

A systematic treatment of processes with stationary nth differences was given by Yaglom [10], who showed that the covariance structure of the nth differences (1) is determined by the one-sided power spectrum of $x(t)$, a measure $\mu(df)$ on the positive frequency axis $f > 0$ satisfying certain finiteness conditions (see (5) below). (The uninteresting situation of a non-ergodic constant component in (1) is excluded here.) In all practical cases, the spectrum consists of a series of Dirac δ functions plus a density component $S(f)$. The straightforward formula

$$D_n(\tau;\mu) = \int_0^\infty [2\sin(\pi f\tau)]^{2n} \mu(df) \qquad (2)$$

(Ref. [10], p. 94) defines a mapping from the measure μ to the function $D_n(\cdot;\mu)$. As indicated above, this mapping is known to be one-to-one for $n = 1$. For $n = 2$, we show now that the mapping is not one-to-one by exhibiting two spectra that map to the same D_2. Let $\mu(df) = f^{-3}df$. Then

$$D_2(\tau;\mu) = 16\pi^2\tau^2 \int_0^\infty x^{-3} \sin^4 x\,dx = 16\pi^2 (\ln 2)\,\tau^2$$

Define a spectrum μ_0 consisting of a series of δ functions: for each integer k, place a δ function of weight 4^{-k} at frequency 2^k. Then

$$D_2(\tau;\mu_0) = 16 \sum_{k=-\infty}^{\infty} \frac{\sin^4(2^k\pi\tau)}{4^k}. \qquad (3)$$

We assert that $D_2(\tau;\mu_0) = 16\pi^2\tau^2$. In view of the identity

$$\sin^4 x = \sin^2 x - \tfrac{1}{4}\sin^2 2x, \qquad (4)$$

the right side of (3) equals

$$16 \lim_{n\to\infty} \left[\sum_{k=-n}^{\infty} \frac{\sin^2(2^k\pi\tau)}{4^k} - \sum_{k=-n}^{\infty} \frac{\sin^2(2^{k+1}\pi\tau)}{4^{k+1}}\right.$$

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The two series cancel except for the one term $4^{-n}\sin^2(2^n\pi\tau)$, whose limit is $\pi^2\tau^2$. This shows that $(\ln 2)\,\mu_0(df)$ and $f^{-3}df$ have the same D_2 and the same Allan variance, a constant for all τ.

An approximation of μ_0 is shown in Figure 1(a), in which the δ function at 2^k is represented by a rectangle of width 2^{k-4} and height proportional to 8^{-k}. Notice that $\mu_0(df)$ has the same coarse power distribution as $f^{-3}df$, in the sense that, for both spectra, the power in any frequency band $[0, b]$ is 4 times the power in $[2b, 2b]$. It turns out that all examples of the spectral ambiguity of D_2 are connected to this property, which is given a precise description below.

Before giving our main results, we introduce some notation and terminology. The actual random processes are in the background; we deal only with their spectra, defined as measures with certain properties. As in Part II, it is convenient to work on a vector space of signed measures. A one-sided signal *spectrum* is defined to be a real-valued (signed) measure μ on the Borel subsets of the positive real axis whose absolute-value measure $|\mu|$ satisfies

$$|\mu|([1,\infty[) < \infty, \quad \int_{]0,1[} f^{2n}\,|\mu|(df) < \infty, \qquad (5)$$

for some nonnegative integer n. A spectrum is said to have *degree* n if n is the least nonnegative integer for which (5) holds. The spectra $f^{-3}df$ and μ_0 both have degree 2. If μ is a signed spectrum of degree $< n$, define $D_n(\tau;\mu)$ by (2). For a positive spectrum, $D_n(\tau;\mu)$ can be interpreted as an actual variance. For signed spectra we still use the term "variance" even though $D_n(\tau;\mu)$ can assume any real value, and $D_2(\tau;\mu)/(2\tau)^2$ is still called Allan variance. If c is a positive real number, define $\mu(c\,\cdot)$ as the signed measure that maps a set A to $\mu(cA) = \mu\{cx : x \in A\}$.

Define the *octave variance* of a signed spectrum μ by

$$V_R(\tau;\mu) = 8\pi^2 \int_{[(4\tau)^{-1},(2\tau)^{-1}[} f^2\mu(df)$$

also called "Rutman's bandpass variance" [11]. As in Part II, this "variance" can assume any real value. The spectra $f^{-3}df$ and μ_0 both have *constant* octave variances. Let μ be a signed spectrum, and let $\nu(df) = 4\pi^2 f^2\mu(df)$. One can prove the equivalence of the following three conditions: i) μ has constant octave variance; ii) $4\mu(2\,\cdot) = \mu$; iii) $\nu(2\,\cdot) = \nu$. If a nonzero μ has constant octave variance, so does $|\mu|$, and their degree is 2. Any spectral density of form $\phi(f)f^{-3}$ where $\phi(f)$ is locally integrable and satisfies $\phi(2f) = \phi(f)$, has constant octave variance. An example is the approximation to μ_0 shown in Figure 1(a). More generally, one can start with any finite real measure μ on an octave interval, and extend it to the positive real axis by condition (ii).

The main results of this letter can now be stated.

THEOREM 1. *Signed spectra of degree < 1 are uniquely determined (1) by their first-difference variances, (2) by their second-difference or Allan variances.*

THEOREM 2. *A signed spectrum has constant Allan variance if and only if it has constant octave variance. In this case, the two variances are equal.*

THEOREM 3. *Two signed spectra have the same Allan variance if and only if they have the same octave variance.*

An example for Theorem 3 is given by the nonnegative spectral densities

$$S_1(f) = f^{-2} + f^{-4}, \quad S_2(f) = S_1(f) - 2f^{-3}\cos(2\pi\log_2 f)$$

(see Figure 1(b)). Their difference is a signed spectrum with octave variance identically equal to zero. By Theorem 2, its Allan variance is also zero. Just enough of this signed spectrum has been subtracted from $S_1(f)$ to produce a null in $S_2(f)$ at $f = 1$. Again, the two spectra have the same coarse distribution of power over the frequency axis.

We now sketch the arguments for these theorems. Complete proofs will be submitted elsewhere. The "if" part of Theorem 2, and the equality of variances, can be proved by generalizing the computation of $D_2(\tau;\mu_0)$. Because of (4) and condition (ii) above, the integral (2) for $D_2(\tau;\mu)$ reduces to

$$16\lim_{\varepsilon\to 0}\int_{[\varepsilon,2\varepsilon[} \sin^2(\pi f\tau)\,\mu(df),$$

which reduces to $2\tau^2 V_R$.

The proofs of the other assertions depend on a function called the *generalized autocovariance* (GACV), defined for a signed spectrum μ of degree n by

$$R(t;\mu) = \int_0^\infty \left[\cos(2\pi ft) - \frac{1}{1+(2\pi f\tau_1)^{2n}}\sum_{j=0}^{n-1}\frac{(-1)^j(2\pi ft)^{2j}}{(2j)!}\right]\mu(df), \quad \text{(i)}$$

where τ_1 is an arbitrary positive number. It can be shown [12] that $R(t;\mu)$ is continuous. This is a form of generalized Fourier transform, tailored for the needs at hand. By straightforward manipulations, one can show that it has the expected linearity and change-of-scale properties, modulo a polynomial.

PROPOSITION 1. *Let μ, μ_1 and μ_2 be signed spectra, c, c_1 and c_2 real numbers, $c > 0$. The functions a) $R(t;c_1\mu_1 + c_2\mu_2) - c_1R(t;\mu_1) - c_2R(t;\mu_2)$ and b) $R(ct;\mu(c\,\cdot)) - R(t;\mu)$ are polynomials.*

It can be shown that μ and R form a Fourier transform pair as functionals on a space of test functions. Let $\mathcal{T}$ be the set of complex-valued functions $\varphi(t)$ whose Fourier transform $\Phi(f)$ is infinitely differentiable and of compact support in the positive real line. Any $\varphi \in \mathcal{T}$ has the following two properties: i) $\lim_{t\to\pm\infty} t^k\varphi(t) = 0$ for all k; ii) $\int\varphi(t)\,t^k dt = 0$ for all nonnegative integral k, i.e., φ kills polynomials.

THEOREM 4. *If μ is a signed spectrum, then*

$$\int_{-\infty}^{\infty} \varphi(t)\, R(t;\mu)\, dt = \frac{1}{2}\int_0^{\infty} \Phi(f)\, \mu(df) \qquad (7)$$

for all $\varphi \in \mathcal{T}$.

From this it follows that $R(t;\mu)$ for a nonzero μ cannot be a polynomial. For if R is a polynomial, then the left side of (7) is zero for all test functions. We state this formally as follows.

THEOREM 5. *If* μ *is a signed spectrum, and* $R(t;\mu)$ *is a polynomial, then* $\mu = 0$. *Consequently,* $R(t;\mu)$ *uniquely determines* μ.

The next theorem shows how to calculate D_n from R. Let δ_τ^2 denote the central difference operator that acts on a function $g(t)$ by $\delta_\tau^2 g(t) = g(t+\tau) - 2g(t) + g(t-\tau)$.

THEOREM 6. *If* μ *is a signed spectrum of degree* $\leq n$, *then*

$$D_n(\tau;\mu) = \left(-\delta_\tau^2\right)^n R(0;\mu). \qquad (8)$$

This result follows from a straightforward application of $\left(-\delta_\tau^2\right)^n$ to the integrand in (6). For $n = 1$ and 2, (8) takes the forms

$$D_1(\tau) = -\delta_\tau^2 R(0) = 2\left[R(0) - R(\tau)\right], \qquad (9)$$

$$D_2(\tau) = \delta_\tau^4 R(0) = 6R(0) - 8R(\tau) + 2R(2\tau), \qquad (10)$$

since $R(t)$ is even. These relationships are well known and easy to derive for a *stationary* process with autocovariance $R(t)$. The present theory applies to processes with stationary nth differences.

Proof of Theorem 1(a). Let μ be the difference of two signed spectra of degree ≤ 1 having the same D_1. Then μ has degree ≤ 1, and $D_1(\tau;\mu) = 0$ for all τ. According to (9), $R(t;\mu)$ is a constant. By Theorem 5, $\mu = 0$.

Theorem 1(b) follows from Theorem 1(a) and Theorem 7, which asserts a formula of Lindsey and Chie [3]. First, we need an asymptotic estimate for D_n.

LEMMA 1. *If* $n \geq 1$ *then* $D_n(\tau;\mu) = o(\tau^{2n})$ *as* $\tau \to \infty$.

THEOREM 7. *If* μ *is a signed spectrum of degree* ≤ 1, *then*

$$D_1(\tau;\mu) = \sum_{k=0}^{\infty} 4^{-k-1} D_2\left(2^k\tau;\mu\right) \qquad (11)$$

Proof. Eqs. (9) and (10) give $D_1(\tau) = [D_2(\tau) + D_1(2\tau)]/4$. Substitute a similar expression for $D_1(2\tau)$, and so on. By Lemma 1, the remainder term $4^{-m}D_1(2^m\tau)$ tends to zero.

Therefore, if μ has degree ≤ 1, then $D_2(\tau;\mu)$ determines $D_1(\tau;\mu)$, which determines μ.

Proof of Theorem 3. For the "if" part, let μ_1 and μ_2 have the same octave variance. Then the octave variance of $\mu = \mu_1 - \mu_2$ is zero. By Theorem 2 "if", $D_2(\tau;\mu) = 0$. For the "only if" part, let μ_1 and μ_2 have the same Allan variance, and let $\mu = \mu_1 - \mu_2$. Then $D_2(\tau;\mu) = 0$. By (10), $R(\tau;\mu) - 4R(\tau/2;\mu)$ is a constant. By Proposition 1(b), $R(\tau/2;\mu) - R(\tau;\mu(2\cdot))$ is a polynomial. Therefore, so is $R(\tau;\mu) - 4R(\tau;\mu(2\cdot))$. By Proposition 1 (c), $R(\tau;\mu - 4\mu(2\cdot))$ is a polynomial. By Theorem 5, $\mu - 4\mu(2\cdot) = 0$; that is, μ has constant octave variance V_R, which, by the parts of Theorem 2 that we know, must be zero because $V_R = D_2$. Therefore, μ_1 and μ_2 have the same octave variance.

Proof of Theorem 2 "only if". If μ has constant Allan variance, then, for some constant c, $cf^{-3}df$ has the same Allan variance as μ. By Theorem 3 "only if", the two signed spectra have the same octave variance, namely, a constant.

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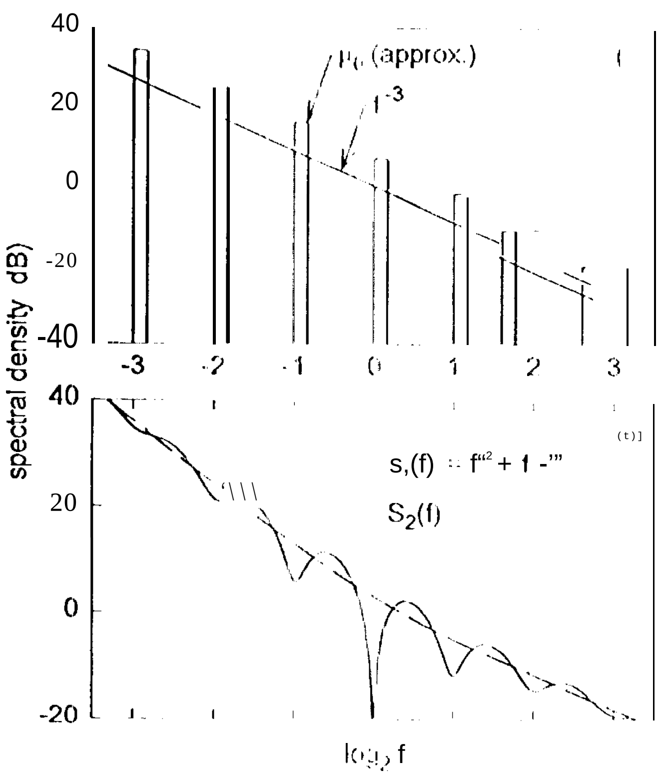

FIG. 1. a) The approximate δ function spectrum has the same constant Allan variance as f^{-3}. b) These two spectra have the same non-constant Allan variance. The spectrum S_2 is obtained from S_1 by adding a signed spectrum with zero Allan variance.