Precision Density Measurements Near the Helium Lambda Transition Using High-Q Microwave Cavities

D. M. Strayer, JPL, Caltech
W. Jiang, N.-C. Yeh, N. Asplund, Caltech

Outline:
- Motivation
- Technical Approach
- Experimental
- Deconvolution Algorithm
  - Summary

Acknowledgement:
- NASA Contract
- Packard Foundation
Abstract Submitted
for the MAR96 Meeting of
The American Physical Society

Sorting Category: 9.c

Precision Density Measurements Near the Helium Lambda Transition Using High-Q Microwave Cavities

D. M. STRAYER, Jet Propulsion Laboratory, Caltech; W. JIANG, N.-C. YEH, N. ASPLUND, Caltech — A new experimental approach for high-precision density measurements of liquid helium near the lambda transition is proposed. Using a high-Q Nb microwave cavity ($Q \sim 10^{10}$) and the high-resolution thermometry (HRT), the changes in the density of helium that fills the cavity can be detected to high precision by accurate measurements of the resonant frequency shift ($\Delta f$) as a function of the temperature. Since the frequency shift provides direct information for the changes in the dielectric constant, and since the dielectric constant is related to the density through the Clausius-Mosotti relation, the capability of high resolution frequency measurements (to one part in $10^{10}$) will enable us to resolve density changes to one part in $10^{-10}$. Numerical calculations have been performed to demonstrate the feasibility of this approach for mapping out the density profile of liquid helium which couples to the TE modes of a microwave cavity. For temperatures very near the lambda transition, a superfluid-normal fluid interface develops inside the cavity. A numerical deconvolution technique is established to resolve the helium density profile in the cavity. Preliminary experimental data using a TM010 niobium cavity and with microkelvin temperature resolutions will be presented.

*Supported by NASA contract and Packard Foundation.

Wen Jiang
wj@cco.caltech.edu
Caltech

Electronic form version 1.1

***END***
Motivation

- Test of the Critical Relation of $\beta_p$ near the Lambda Transition

$$\beta_p = A|t|^{-\alpha}(1 + D|t|^z + \cdots) + B$$

$$\equiv -\left. \frac{1}{\rho} \frac{\partial \rho}{\partial T} \right|_p$$

High precision measurement of $\rho(T)$

$$\Rightarrow \beta_p(T) \Rightarrow \alpha, A, B, D, x$$

- Related Experimental Work:

| Physical Quantity         | Sensitivity $(\epsilon_\lambda \equiv |1 - (T/T_\lambda)|)$ |
|---------------------------|---------------------------------------------------------------|
| Capacitor (C)             | $\delta C/C \sim 10^{-9}, \epsilon_\lambda \sim 10^{-5}$    |
| Van Degrift & Pellam      | 1974                                                          |
| thermal expansion coefficient ($\beta_p$) | $\delta P \sim 0^{-7}$ bar, $\epsilon_\lambda \sim 10^{-7}$ |
| Muellel, Ahlers, & Pobell | 1976                                                          |
| heat capacity (C)         | $\delta Q/Q \sim 10^{-4}, \epsilon_\lambda \sim 10^{-9}$    |
| Lips, et al (to be published in PRL) |                                                     |
| resonant frequency        | $\delta f/f_0 \sim 10^{-13}, \epsilon_\lambda \sim 10^{-9}$ |
| shift ($\Delta f/f_0$)    |                                                               |

- Other Applications:

$^3\text{He} - ^4\text{He}$ mixture: $\rho_{mix} = x\rho_{^3\text{He}} + (1 - x)\rho_{^4\text{He}}$

Precision measurements of $\rho \Rightarrow$ precision measurement of $x$. 
Technical Approach

- Clausius-Mossotti Relation \((\varepsilon \leftrightarrow \rho)\)

\[
\varepsilon + 2 = \frac{4\pi \alpha_0}{3M}\rho
\]

\(\varepsilon\) : dielectric constant

\(\alpha_0\) : polarizability

(assumed to be constant near the lambda transition)

\(M\) : molecular weight

\(\rho\) : density

- In a microwave cavity, the electric fields couple to the dielectric constants \(\varepsilon\)

- Cavity resonant frequency shift \((\Delta f)\) due to the small change of \(\varepsilon\) of liquid helium:

\[
\frac{\Delta f}{f_0} = \frac{f - f_0}{f_0} \frac{\int_{V_0} (\varepsilon - \varepsilon_0) |E_0|^2 dV}{\int_{V_0} \varepsilon_0 |E_0|^2 dV}
\]

\(f_0\) : resonant frequency of the resonator at a reference temperature

\(V_0\) : volume of the cavity

\(E_0\) : the electric field of the resonant mode

\(\varepsilon_0\) : dielectric constant(s) at the reference temperature
Consider a cylindrical cavity maintained at a uniform temperature $T$,

$$T(0) + T''(0) \gamma_0 z$$

$$\gamma_0 = 1.273 \times 10^{-6} \text{K/cm}$$


If $T(0) < T < T(d)$, a superfluid/normal fluid interface appears inside the cavity at the position

$$z_0 = \frac{T - T(0)}{\gamma_0}.$$
Deconvolution for Resolving $\rho(z,T)$ near the Lambda Transition

$$ \frac{\Delta f(T)}{f_0} \quad \Rightarrow \quad \varepsilon(z,T) \quad \Rightarrow \quad \rho(z,T) $$

Clausius-Mossotti relation

For $TE_{nm\ell}$ mocks,

$$ \vec{E} = E_\theta \hat{e}_\theta, \quad E_\theta = \frac{ik\eta aH_0}{\bar{P}_{nm}^\prime} \cdot \left( \frac{\bar{P}_{nm}^\prime r}{a} \right) \sin \left( \frac{\ell \pi z}{d} \right) $$

$$ \frac{\Delta f(T)}{f_0} = \frac{\int_0^d [\varepsilon(z,T) - \varepsilon(z, T_\lambda(0))] \sin^2 \left( \frac{\ell \pi z}{d} \right) \, dz}{\int_0^d \varepsilon(z, T_\lambda(0)) \sin^2 \left( \frac{\ell \pi z}{d} \right) \, dz} $$
Experimental

—— High-Q Nb microwave cavity:

\[ Q \sim 10^9 \text{ near } 2.2 \, \text{K} \]

—— High-Resolution Thermometry (HRT):

**SQUID** magnetic susceptibility:


Temperature resolution: \( \sim 10^{-10} \, \text{K} \)

Temperature stability: \( \sim 10^{-9} \, \text{K} \)

—— High-resolution frequency source:

**HP** high-stability source: \( \delta f/f \sim 10^{-13} \)

Phase-lock loop technique: \( \delta f/f \sim 10^{-15} \)
Block Diagram of Resonant Frequency Measurement Using a High-Q Microwave Cavity
Schematic Measurement Cell Design
Deconvolution Algorithm

For $T_\lambda(0) < T < T_\lambda(z)$, a given temperature $T \mapsto$ an interface position $z_0$

Choose

$$T_j = T_\lambda(0) + \gamma_0 z_j, j = 1, ... N,$$

such that $z_j = j \frac{d}{N}$.
Discretization

\[
\frac{\Delta f}{f_0}(T) = \frac{f_0}{T_0} = \frac{\int_0^d [\varepsilon(z, T) - \varepsilon(z, T_\lambda(0))] \sin^2 \left( \frac{l \pi z}{d} \right) dz}{\int_0^d \varepsilon(z, T_\lambda(0)) \sin^2 \left( \frac{l \pi z}{d} \right) dz}
\]

\[
\Rightarrow \left| 1 - \frac{\Delta f}{f_0}(T_j) \right| \sum_{i=1}^N [\varepsilon(t_i^0) - \varepsilon_0] \sin^2 \left( \frac{l \pi z_i}{d} \right) - \sum_{i=1}^N [\varepsilon(t_i^0) - \varepsilon_0] \sin^2 \left( \frac{l \pi z_i}{d} \right)
\]

\[
= \frac{\Delta f \varepsilon_0 N}{f_0} \frac{2}{2}
\]

where

\[
\varepsilon_0 \equiv \varepsilon(T = T_\lambda),
\]

\[
t_i^j = t(T_j, z_i) \equiv T_j - T_\lambda(z_i) = (j - i) \frac{\gamma_0 d}{N}
\]

\[
z_i = i \frac{d}{N}
\]

\[
i = 1, ..., N, j = 1, ..., N
\]
\(
\implies \text{Linear algebra problem}
\)

\[
A \mathbf{x} = \mathbf{b} \implies \mathbf{x} = (A^T A)^{-1} A^T \mathbf{b}
\]

where

\[
\mathbf{x} = \varepsilon - \varepsilon_0, \quad \mathbf{b} = \frac{\varepsilon_0 N}{2} \mathbf{C}
\]

\[
\varepsilon = \left(\begin{array}{c}
\varepsilon(-dt) \\
\varepsilon\left(-\frac{N-1}{N} dt\right) \\
\vdots \\
\varepsilon(dt)
\end{array}\right), \quad \mathbf{C} = \left(\begin{array}{c}
\frac{\Delta f_0}{f_0}(T_1) \\
\frac{\Delta f_0}{f_0}(T_2) \\
\vdots \\
\frac{\Delta f_0}{f_0}(T_N)
\end{array}\right)
\]

\[
A = \left(\begin{array}{cccc}
(1-C_1)\mathbf{u}^T & 0 & \cdots & 0 \\
(1-C_2)\mathbf{u}^T & 0 & \cdots & 0 \\
\vdots & 0 & \cdots & 0 \\
(1-C_N)\mathbf{u}^T & 0 & \cdots & 0
\end{array}\right) - \left(\begin{array}{cccc}
0 & \mathbf{u}^T & 0 & \cdots & 0 \\
0 & 0 & \mathbf{u}^T & \cdots & 0 \\
\vdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & \mathbf{u}^T
\end{array}\right), \quad \mathbf{u} = \left(\begin{array}{c}
\sin^2\left(\frac{\ell \pi z_1}{d}\right) \\
\sin^2\left(\frac{\ell \pi z_2}{d}\right) \\
\vdots \\
\sin^2\left(\frac{\ell \pi z_N}{d}\right)
\end{array}\right)
\]

With \( z? = 1 \& 3,2N \) unknowns: \( \varepsilon - \varepsilon_0 \) (assume \( \varepsilon_0 \) is known), and 2N equations.
Computer Simulation

\begin{align*}
\rho(t) &= 0.146081 - 0.1451189 \times 10^{-6.0} \times (2630.9 t \\
&- 7768.2 \, t \log_{10}|t| + 20547.1 \, t^2 + 15963.2 \, t \log_{10}|t| \\
&+ 24525.3 \, t^3 - 12728 \, t^2 \log_{10}|t|), \text{ for } t < 0;
\end{align*}

\begin{align*}
\rho(t) &= 0.146081, t = 0;
\end{align*}

\begin{align*}
\rho(t) &= 0.146081 + 0.1451189 \times 10^{-6.0} \times (-31288.6 \, t \\
&- 8086.1 \, t \log_{10}|t|) + 0738.1 \, t^2 - 3010.4 \, t^3), \text{ for } t > 0.
\end{align*}

where \( t \equiv T - T_\lambda(z) \)

\( \alpha_0 = 0.123363N_A = 0.204849 \, \text{cm}^3/\text{molecule} \)

\( M = 6.6424 \times 10^{-24} \, \text{g} \)

\( \epsilon(t) = \frac{1 + 2a \rho}{1 - a \rho}, \quad a \equiv \frac{4\pi\alpha_0}{3M} \)

\( \frac{\Delta f}{f_0}(T) = \frac{f - f_0}{f_0} = \frac{\int_0^d [\epsilon(z, T) - \epsilon(z, T_\lambda(0))] \sin^2(\frac{\ell \pi z}{d}) \, dz}{\int_0^d \epsilon(z, T_\lambda(0)) \sin^2(\frac{\ell \pi z}{d}) \, dz} \)

*Refs: Dommelweg, Kriegsmann & Barenghi, "The Direction Properties of Liquid Helium at its Saturation Vapor Pressure", 1983*
$T = T_\lambda(0)$

$T_\lambda(0) < T < T_\lambda(d)$

$T \neq T_\lambda(d)$
\[ \frac{\delta f}{f} \sim 10^{-13}, \quad \frac{\delta T}{T} \sim 10^{-9} \]
\[
\frac{\delta f}{f} \sim 10^{-13}, \quad \frac{\delta T}{T} \sim 10^{-9}
\]
\[ \delta f \approx 10^{-13}, \quad \delta T \approx 10^{-11} \]
\[ \frac{\delta T}{T} \sim 10^{-9}, \quad \frac{\delta f}{f} \sim 10^{-13} \]
\[ \frac{\delta f}{f} \sim 10^{-13}, \quad \frac{\delta T}{T} \sim 10^{-11} \]
Expected Resolution in Density Measurement

<table>
<thead>
<tr>
<th></th>
<th>$\delta f/f$</th>
<th>$\delta T/T$</th>
<th>$6p/p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>earth-bound laboratory</td>
<td>$\sim 10^{-13}$</td>
<td>$\sim 10^{-9}$</td>
<td>$&lt; 3.5 \times 10^{-10}$</td>
</tr>
<tr>
<td></td>
<td>$\sim 10^{-13}$</td>
<td>$\sim 10^{-11}$</td>
<td>$&lt; 0.8 \times 10^{-10}$</td>
</tr>
<tr>
<td>microgravity environment</td>
<td>$\sim 10^{-13}$</td>
<td>$\sim 10^{-9}$</td>
<td>$&lt; 2.5 \times 10^{-10}$</td>
</tr>
<tr>
<td></td>
<td>$\sim 10^{-11}$</td>
<td>$\sim 10^{-11}$</td>
<td>$&lt; 2.5 \times 10^{-12}$</td>
</tr>
</tbody>
</table>
Summary

- **High precision** measurements of helium density near the *lambda* transition

  High-Q Nb cavity (*Q ~ 10^9* near 2.2 K)
  High-resolution thermometry (*|T - T_\lambda| ~ 10^{-9} K*)
  High-resolution frequency source (*\delta f/f \sim 10^{-13}*)

  \[ \Rightarrow \delta \rho/\rho \sim 10^{-10} \]

- **Deconvolution** algorithm

  \[ \frac{\Delta f}{f_0}(T) \Rightarrow \epsilon(T, z) [\Rightarrow \rho(T, z)] \]

*Potential of the Technique:*

  Higher precision density measurements in the microgravity environment:
  \[ \delta T/T \sim 10^{-11} \Rightarrow \delta \rho/\rho \sim 10^{-12} \]