Remark on Algorithm 723: Fresnel Integrals

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ACM TOMS *Algorithm 723: Fresnel Integrals* has been improved to provide more precise results for x >>0.

Categories aod Subject Descriptors: G.1.2 [Numeric-al **Analysis**]: Approximation- *rational approximation*; G.4 [**Mathematics of Computing**]: Math ematical Software *certification and testing*

Additional Key Words and Phrases: Special functions

Concerning evaluation of Fresnel Cosine and Sine integrals $C(x)$ and $S(x)$ for $x > 1.6$, the last sentence in the description of Algorithm 723 [Snyder 1993] states "Extensive accuracy testing would simply have validated the trigonometric function routines."

Alfred H. Morris, Jr. has correctly pointed out that the procedures described in [Snyder 1 993] to compute $C(x)$ and $S(x)$ for $x > 1.6$ require computing $\sin \pi x^2/2$ and $\cos \pi x^2/2$. Thus the quoted statement begs the question whether one might compute these trigonometric functions more accurately than can be done by naïve application of the Fortran intrinsic trigonometric function routines, when x is large.

The answer is in the affirmative. By separating into its integer and fractional parts one can compute these functio ns with substantially better accuracy.

Let $x = n + f$ where $n = \lfloor x \rfloor$, and let $n = 2k + j$, where j is $n \bmod 2$. Then $\sin \pi x^2/2 = \sin \pi(4k^2 + 4kj + j^2 + 4kf + 2jf + f^2)/2$. By application of simple trigonometric identities, one observes that. this is equal to $\sin \pi(2kf + f^2/2)$ or $\cos \pi(2kf + f + f^2/2)$, depending on whether j is zero or one, respectively. Letting $kf = m + g$, where $m = \lfloor kf \rfloor$, $\sin \pi x^2/2$ becomes $\sin \pi(2g + f^2/2)$ or $\cos \pi(2g + j + f^2/2)$, depending on whether j is zero or one, respectively. A similar development may be used to improve computation of $\cos \pi x^2/2$.

in the procedures of Algorithm 723, we have replaced usage of Fortran intrinsic trigonometric function routines to calculate $\sin \pi x^2/2$ and $\cos \pi x^2/2$ by usage of

The work described in this paper was carried out at the Jet Propulsion Laboratory, California Institute of Technology, under contract with the National Aeronautics and Space Administration.
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routines that implement these ideas. A revised algorithm has been submitted to the ACM Algorithm Distribution Service.

1. TESTING

We tested the accuracy of the single precision procedures of Algorithm 723, and the procedures of the revised algorithm, in computing $C(10^x)$ and $S(10^x)$, by subdividing each of the ranges $0 \leq x \leq 6$, $6 \leq x \leq 12$, and $12 \leq x \leq 18$ into 20000 equal intervals, choosing a value of x from a uniform pseudo-random distribution over each interval, and comparing the result to one computed by the double precision procedures of Algorithm 723.

Test values were calculated using IEEE format 32-bit floating point numbers; reference values were calculated using IEEE format 64-bit floating point numbers.

Let A be absolute error, U the error measured in ULP, E the exponent of the floating-point result, f the fraction of the floating-point result (assumed to be normalized), r the radix of floating-point numbers and ν the number of base-r digits in f. Then $U = A/r^{E-\nu}$. The more familiar relative error R (in multiples of r^{ν}) = U/f. Since in a normalized base-r floating-point number, $1/r \leq f < 1$, $U \leq R < rU$.

In the following tabular summaries, the heading *b bits incorrect* indicates the number of low-order bits of the result that were incorrectly computed. $b = \lceil \log_2 U \rceil$.

Summary of errors committed by $C(x)$ in Algorithm 723, as published								
	Percent of samples with b bits incorrect							
Range b:	$\frac{1}{2}$	1	2	3	4	5	6	7
$0 \leq x \leq 6$	39.0	25.9	21.0	10.7	3.2	0.3	0	
$6 \leq x \leq 11$	18.7	15.2	20.5	22.7	16.7	5.8	0.3	0
$12 \leq x \leq 18$	13.1	10.4	15.6	21.2	22.6	14.0	3.1	0

Summary of errors committed by $S(x)$ in Algorithm 723, as published								
	Percent of samples with b bits incorrect							
Range b:	$\frac{1}{2}$	1	2	3	4	5	6	7
$0 \leq x \leq 6$	35.7	25.5	24,0	11.5	3.1	0.2	0	
$6 \leq x \leq 12$	18.8	14.8	*20.1*	23.6	16.6	5.8	0.4.4	0
$12 \leq x \leq 18$	1?.7	10.8	15.6	21.9	22.8	13.2	33.0	0

Summary of errors committed by revised Algorithm 723										
	Percent of samples of $C(x)$ with b bits incorrect					Percent of samples of $S(x)$ with b bits incorrect				
Range b:	$\frac{1}{2}$	1	2	3	4	$\frac{1}{2}$	1	2	3	4
$0 \leq x \leq 6$	63.2	28.7	7.4	0.6	0	60.4	29.1	9.6	0.9	0
$6 \leq x \leq 12$	73.8	25.5	0.7	0		73.7	25.6	0.7	0	
$12 \leq x \leq 18$	74.3	25.5	0.2	0		74.8	25.0	0.3	0	

Clearly, the revised procedures provide a substantial improvement.

The modifications described here allow us to change the formula $|x| > 2.0/\rho^{1/2}$ in the last paragraph on page 453 of [Snyder 1993] to $|x| > 2.0/\rho$.

ACKNOWLEDGMENTS

We wish to thank Alfred H. Morris, Jr. for bringing the possibility of this improvement to our attention, and for approving the use of the routines SIN1 and COS1 from [Morris 1993], for computing $\sin \pi x$ and $\cos \pi x$.

References

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