

QUANTUM RESONANCE APPROACH TO COMBINATORIAL OPTIMIZATION

Michail Zak

Ultracomputing Group, Mail Stop 525-3660
Jet Propulsion Laboratory
California Institute of Technology
Pasadena, CA 91109-8099

Abstract

It is shown that quantum resonance ~~can be used for combinatorial~~ optimization. The advantage of the ~~approach is in independence of the~~ computing time upon the dimensionality ~~of the problem.~~ As an example, the solution to a constraint satisfaction ~~problem of exponential complexity is~~ demonstrated.

Combinatorial problems are among the hardest in theory of computations. They include a special class of so called **NP-complete** problems which are considered to be intractable by most of the theoretical computer scientists. A typical representative of this class is a famous traveling-salesman problem (TSP) of determining the shortest closed tour that connects a given set of n points in the plane. As for any of **NP-complete** problem, here the algorithm for solution is very simple: enumerate all the tours, compute their lengths, and select the shortest one. However, the number of tours is proportional to $n!$ and that leads to exponential growth of computational time as a function of the dimensionality n of the problem, and therefore, to computational intractability.

It should be noticed that, in contradistinction to continuous optimization problems where the knowledge about the length of a trajectory is transferred to the neighboring trajectories through the gradient, here the gradient does not exist, and there is no alternative to a simple enumeration of tours.

The class of **NP-complete** problems has a very interesting property: if any single problem (including its worse case) can be solved in polynomial time, then every **NP-complete** problem can be solved in polynomial time as well. But despite that, there is no progress so far in removing a curse of combinatorial explosion: it turns out that if one manages to achieve a polynomial time of computation, then the space or energy grow exponentially, i.e., the effect of combinatorial explosion stubbornly reappears. That is why the intractability of **NP-complete** problems is being observed as a fundamental principle of theory of computations which plays the same role as the second law of thermodynamics in physics.

At the same time, one has to recognize that the theory of computational complexity is an attribute of a digital approach to computations, which means that the monster of NP-completeness is a creature of the Turing machine.

As an alternative, one can turn to an analog device which replaces digital computations by physical simulations. Indeed, assume that one found such a physical phenomenon whose mathematical description is equivalent to those of a particular NP-complete problem. Then, incorporating this phenomenon into an appropriate analog device one can simulate the corresponding NP-complete problem. In this connection it is interesting to note that, on the first sight, all the NP-complete problems are fundamentally different from natural phenomena: they look like man-made puzzles while their formal mathematical framework is mapped into decision problems with yes/no solutions. However, one should recall that physical laws can also be stated in a “man-made” form: The least time (Fermat), the least action (in modifications of Hamilton, Lagrange, or Jacobi), and the least constraints (Gauss).

In this note we will concentrate attention to mapping a constraint satisfaction problem into physical phenomena of quantum resonance on a conceptual level, without going into details of actual implementations.

Consider a quantum system subject to a small white-noise perturbing interaction, and let the perturbation be switched on at zero time. The Hamiltonian of the system can be presented as a sum of the time-independent and oscillating components:

$$H = H_0 + \varepsilon_0 H_1 \delta(\omega t), \quad \varepsilon_0 \ll 1 \quad (1)$$

where H_0 and H_1 are constant Hermitian matrices, ω is the frequency of perturbations, and $\delta(\omega t)$ is the Dirac function.

The probability of a transition from state k to q is proportional to the product^[1].

$$P_{kq} \propto \left| \langle \varphi_k^* | H_1 | \varphi_q \rangle \right|^2 \left[\frac{\text{Sin} \frac{1}{2} (a_{kq} - \omega) t}{a_{kq} - \omega} \right]^2 \quad (2)$$

Here φ_j are the eigenstates of H_0 :

$$H_0\varphi_j = E_j\varphi_j \quad j = 1, 2, \dots, N \quad (3)$$

where E_j are the energy eigenvalues,

$$a_{kq} = E_k - E_q, \quad k, q = 1, 2, \dots, N \quad (4)$$

The resonance, i.e., a time-proportional growth of the transition probability P_{kq} occurs when $\omega = \beta_{kq}$. But since all the frequencies ω have the same energy contribution (see Eq. (1)), the decisive factor for the choice of a certain transition will be the time independent multiplier in Eq. (2). Hence, those two states k and q for which

$$|\varphi_k^* H_1 \varphi_q| = \max_{\substack{i=1, \dots, N \\ j=1, \dots, N}} |\varphi_i^* H_1 \varphi_j| \quad (5)$$

will have the highest probability to be detected.

The property (5) can be mapped into several computational problems, and, for the purpose of illustration, we will choose the following one: given n different items to be distributed over n different places; the cost of an β^{th} item put in a γ^{th} place is $\lambda_{\beta}^{(\gamma)}$. Find such a distribution that the total cost is equal (or the closest) to a prescribed number a^2 .

This problem is typical for optimal design, and it represents a so called design to cost approach. Since the cost of a particular distribution is expressed by the sum

$$E_j = \sum_{\beta=1}^n \lambda_{\beta}^{(\gamma)}, \quad j = 1, 2, \dots, N = n^n \quad (6)$$

one has to compute all the n^n sums (6) in order to find those E_k which is equal (or the closest) to a , i. e.,

$$E_k = \alpha^2, \text{ or } E_k - \alpha^2 \Big| = \min_{k=1 \dots N} E_j - \alpha^2 \quad (7)$$

Eq. (7) represents a global constraint, and therefore our problem belongs to the class of so called constraint satisfaction problems which are the hardest among other optimization problems. The constraint (7) prevents one from partitioning the solution into smaller-size sub-problems, and therefore, the problem is **NP-complete**: its computational cost grows exponentially with the growth of the dimensionality n .

Now we will show how this problem can be solved by the quantum device described above in one computational step.

First, let us represent the unitary matrix U_0 corresponding to the time-independent Hamiltonian

$$U_0 = e^{iH_0} \quad (8)$$

as a direct product of n diagonal unitary matrices of the size $n \times n$:

$$U_0 = U_1 \otimes U_2 \otimes \dots \otimes U_n \quad (9)$$

while

$$U_\gamma = \begin{pmatrix} e^{i\lambda_1^{(\gamma)}} & \dots & \mathbf{0} \\ \vdots & \dots & \vdots \\ \mathbf{0} & \dots & e^{i\lambda_n^{(\gamma)}} \end{pmatrix} \quad (10)$$

Then the unitary matrix U_0 in (9) will be also diagonal

and

$$H_0 = \begin{pmatrix} E_1 & \dots & \mathbf{0} \\ & \dots & \vdots \\ 0 & \dots & E_N \end{pmatrix} \quad N = n^n \quad (11)$$

while E_j is expressed by Eq. (6).

Hence, if one select $\lambda_{\beta}^{(\gamma)}$ in (10) as the costs of a β^{th} item put in a γ^{th} place, then the eigenstates E_j of the Hamiltonian H_0 will represent costs of all $N = n^n$ possible distributions (6).

Now we have to choose the second component of the Hamiltonian, i.e., H_1 (see Eq. (1)) such that the highest probability (5) includes the eigenstate E_k which satisfies the constraint (7). For that purpose assume that

$$H_1 = \left(H_0 - \frac{1}{2\alpha^2} H_0^2 \right) P \left(H_0 - \frac{1}{2\alpha^2} H_0^2 \right) \quad (12)$$

where P is the $N \times N$ idempotent matrix:

$$P = \frac{1}{N} \begin{pmatrix} 1 & \dots & 1 \\ \dots & \dots & \dots \\ 1 & \dots & 1 \end{pmatrix} \quad (13)$$

Then, with reference to Eq. (3), one obtains:

$$P_{kq} \propto E_k - \frac{1}{2\alpha^2} E_k^2 \quad E_q - \frac{1}{2\alpha^2} E_q^2 \quad (14)$$

Thus, the probability P_{kq} has its maximum when

$$E_k = a^2, \text{ and } E_q = a^2 \quad (15)$$

However, since E_j in Eq. (6) take only discrete values which may not commensurate with the prescribed value a^2 , we will turn to Eq. (7) and replace (15) by the following:

$$|E_k - a^2| |E_q - a^2| = \min_{\substack{i=1, \dots, N \\ j=1, \dots, N}} |E_k - a^2| |E_q - a^2| \quad (16)$$

Thus, two states, k and q, have the highest probability to be detected if their eigenvalues E_k and E_q respectively, are the closest to the prescribed value a^2 . Hence, in one (or few) measurement these two states can be detected, and the problem is solved regardless of its dimensionality n.

Other computational mappings can be achieved by changing the representations of (10) and (12).

The space required to perform computations described above grows proportionally only to n^2 because of direct product representation of the time-independent Hamiltonian (see Eqs. (8) and (9)). The second component of the Hamiltonian, i.e., H_1 , (see Eqs. (8), (12) and (13)) is representable in a similar way. The matrix in (13) (associated with a projector operator), is factorizable into the direct product of the N-vector (1,1,...,1) and its conjugate. The last vectors, in turn, can be factorized into the direct product of n-vectors of the same structure.

Thus, it has been demonstrated how a man-made problem of exponential computational complexity which cannot be handled by algorithmic methods, is solved by exploiting a strongly pronounced physical phenomena: quantum resonance.

The research described in this paper was performed by the Center for Space Microelectronics Technology, Jet Propulsion Laboratory, California Institute of Technology and was sponsored by the National Aeronautics and Space Administration, Office of Space Access and Technology.

References

1. D. R. Bates, Quantum Theory, Acord. Press, NY, 1961, p 271