INCOMPATIBLE STOCHASTIC PROCESSES AND COMPLEX PROBABILITIES

Michail Zak
Center for Space Microelectronics Technology
Jet Propulsion Laboratory
California Institute of Technology
Pasadena. CA 91109

Abstract

The definition of conditional probabilities is based upon the existence of a joint probability. However, a reconstruction of the joint probability from given conditional probabilities imposes certain constraints upon the latter, so that if several conditional probabilities are chosen arbitrarily, the corresponding joint probability may not exist. Such an incompleteness in conditional probabilities can be eliminated by introducing complex probabilities.

Physical meaning of the new mathematical formalism, as well as its relation to quantum probabilities, is discussed.

One of the oldest and still unsolved problems in the field of multivariate stochastic processes is to reconstruct a joint probability from several correlated conditional probabilities. This problem has been discussed in [1] - [3]. Its origin is in the fact that classical probability theory defines conditional probabilities based upon the existence of a joint probability. At *the* same time, one can observe correlated stochastic processes which are represented by conditional probabilities. And then the inverse problem of reconstructing the underlying joint probability arises.

As an illustration to this point, consider two coupled diffusion equations:

$$\frac{\partial \rho_1(x_1,x_2)}{\partial t} = D_1(x_2)\frac{\partial^2 \rho_1(x_1,x_2)}{\partial x_1^2}, \quad \frac{\partial \rho_2(x_1,x_2)}{\partial t} = D_2(x_1)\frac{\partial^2 \rho_2(x_1,x_2)}{\partial x_2^2} \tag{1}$$

with the initial conditions:

$$\rho_i(x_i,t'|x_i',t') = \delta(x_i - x_i'), \quad i = 1,2 \tag{2}$$

The solution for $t > t'$ reads:

$$\rho_1(x_1, x_2) = \frac{1}{\sqrt{4\pi D_1(x_2)(t-t')}} \exp\left(-\frac{(x_1 - x_1')^2}{4D_1(x_2)(t-t')}\right) \quad (3)$$

$$\rho_2(x_1, x_2) = \frac{1}{\sqrt{4\pi D_2(x_1)(t-t')}} \exp\left(-\frac{(x_2 - x_2')^2}{4D_2(x_1)(t-t')}\right) \quad (4)$$

Obviously, ρ_1 and p_2 *are* interpreted as the density of x_1 given x_2 and the density of x_2 given x,, respectively, i.e., they represent conditional probabilities:

P1=.fl(41X2), *P2* = $f_2(x_2|x_1)$ *(5)*

As will be shown below, a joint probability for (3) and (4) exists only for special choices of the diffusion coefficients $D_1(x_2)$ and $D_2(x_1)$ when the conditional probabilities are compatible. In order to obtain the compatibility equation, start with the joint probability $\phi(x_1, x_2)$ and recall that

$$\Phi(x_1, x_2) = f_1(x_1|x_2) \int_{-\infty} \Phi(z, x_2)\,dz = f_2(x_2|x_1) \int_{-\infty} \Phi(x_1, z)\,dz \quad (6)$$

whence

$$\frac{f_1(x_1|x_2)}{f_2(x_2|x_1)} = \frac{\int_{-\infty} \Phi(x_1, z)\,dz}{\int_{-\infty} \Phi(z, x_2)\,dz} \quad (7)$$

i.e.

$$\ell n \frac{f_1(x_1|x_2)}{f_2(x_2|x_1)} \; \ell n \int_{-}^{\infty} \Phi(x_1, z)\,dz - \ell n \int_{-\infty} \Phi(z, x_2)\,dz \quad (8)$$

and therefore

$$\frac{\partial^2}{\partial x_1 \partial x_2} \ln \frac{f_1(x_1|x_2)}{f2\ (X21X1)} \equiv 0 \tag{9}$$

Thus the existence of the joint probability density $\Phi(x_1, x_2)$ requires that the conditional probability densities must satisfy the compatibility equation (9). But it is easily verifiable that the solutions (3) and (4) do not satisfy this equation, i.e., they are incompatible:

$$ink(f_1, f_2) = \frac{\partial^2}{\partial x_1 \partial x_2} \ell n \frac{f_1}{f_2} \not\equiv 0 \tag{lo}$$

if

$D_1(x_2)$ and $D_2(x_1)$ are chosen arbitrarily.

At the same time, there is nothing wrong with these solutions since they describe two stochastic processes which can be observed and even implemented by dynamical simulations, [4,5]. Hence, the only conclusion which can be made is that the joint probability in this particular case does not exist! But how “particular” is this case? Based upon the degree of arbitrariness to which the diffusion coefficients can be assigned, it is obvious that the incompatibility of the conditional probabilities is a rule rather than an exception. In other words, there is a class of coupled stochastic processes for which joint probability does not exist, and therefore, they are inseparable, i.e., there is no such transformation of variables which would break them down into independent components.

As known from classical mechanics, the incompatibility conditions are usually associated with a fundamentally new concept or a physical phenomenon. For instance, incompatibility of velocities in fluid

$$\frac{\partial \upsilon_x}{\partial y} - \frac{\partial \upsilon_y}{\partial x} = \text{Curl } \upsilon \neq 0 \tag{11}$$

introduces vorticity and rotational flows, and incompatibility of strains describes continua with dislocations, [6].

In this context one may ask what is a "hidden" physical effect behind the incompatibility (10)? Maybe this incompatibility can be eliminated in the class of complex functions?

Following the formalism of quantum probabilities, let us introduce a complex joint density "amplitude":

$$f(x_1,x_2)=a(x_1,x_2)+i\,b(x_1,x_2) \qquad (12)$$

Then the marginal density "amplitudes" are:

$$f_1(x_1)=\int_{-\infty} a(x_1,x_2)\,dx_2+i\int_{-\infty} b(x_1,x_2)\,dx_2=a_1(x_1)+ib_1(x_1) \qquad (13)$$

$$f_2(x_2)=\int_{-\infty} a(x_1,x_2)\,dx_1+i\int_{-\infty} b(x_1,x_2)\,dx_1=a_2(x_2)+ib_2(x_2) \qquad (14)$$

where

$$a_1(x_1)=\int_{-\infty} a(x_1,x_2)\,dx_2,\; a_2(x_2)=\int_{-\infty}^{\infty} a(x_1,x_2)\,dx_1$$

$$b_1(x_1)=\int_{-\infty} b(x_1,x_2)\,dx_2,\; b_2(x_2)=\int_{-\infty}^{\infty} b(x_1,x_2)\,dx_1 \qquad (15)$$

Following the formalism for conditional probabilities, the conditional density "amplitudes" will be defined as:

$$f_{1|2}=\frac{f(x_1,x_2)}{f_2(x_2)}=\frac{a(x_1,x_2)+ib(x_1,x_2)}{a_2(x_2)+ib_2(x_2)}=\frac{aa_2+bb_2}{a_2^2+b_2^2}+i\,\frac{a_2b-ab_2}{a_2^2+b_2^2}$$

$$(16)$$

$$f_{2|1}=\frac{f(x_1,x_2)}{f_1(x_1)}=\frac{a(x_1,x_2)+ib(x_1,x_2)}{a_1(x_1)+ib_1(x_1)}=\frac{aa_1+bb_1}{a_1^2+b_1^2}+i\,\frac{a_1b-ab_1}{a_1^2+b_1^2}$$

Now we have to depart from the quantum formalism and define the physical probability density as a real part of the "amplitude" in Eqs. (13)-(1 3) requiring that

$$\int\int_{-\infty}^{\infty} a(x_1,x_2)\,dx_1 dx_2 = 1 \tag{17}$$

This condition can be enforced by introducing a normalizing multiplier in (12) which will not effect the conditional probabilities (16).

Now our original problem can be reformulated in the foillowing manner: given two correlated stochastic processes (for instance, (3) and (4)); considering them as real parts of (unknown) complex density amplitudes (16), find the corresponding complex joint density amplitude (12), and therefore, all the marginals (13) and (14), as well as the imaginary parts of conditional density "amplitudes,"

In this case one arraives at two coupled integral equations with respect to $a(x_1, x_2)$ and $b(x_1, x_2)$,

$$\rho_1(x_1,x_2) = \frac{aa_2 + bb_2}{a_2^2 + b_2^2}, \quad \rho_2(x_1,x_2) = \frac{aa_1 + bb_1}{a_1^2 + b_1^2} \tag{18}$$

This system is nonlinear, and very little is known about general properties of its solution. However, cases when a solution does not exist are exceptions rather than a rule. Indeed, discretizing the functions in (18) and replacing the integrals by the corresponding sums (see Eqs. (15)), one reduces (18) to a system of n algebraic equations (with quadratic nonlinearities) with respect to n unknowns. This means that for any arbitrarily chosen conditional densities

$$\rho_1(x_1,x_2) = \mathrm{Re}\, f_{1/2}(x_1,x_2), \quad \rho_2(x_1,x_2) = \mathrm{Re}\, f_{2/1}(x_1 x_2) \tag{19}$$

the system (18) defines two corresponding functions $a(x_1,x_2)$ and $b(x_1,x_2)$ which forma complex joint density "amplitude" (12).

At this point we are ready to formulate the main difference between the complex probabilities introduced above, and those in quantum mechanics: here the imaginary components are defined by incompatibilities (in a sense of Eq. (10)) between correlated stochastic processes, while quantum probabilities are postulated in complex form to explain the wave properties of particles observed in experiments.

Turning to Eq. (16) let us derive the complex version of the Bayes' rule:

$$\operatorname{Re} f_{1|2} \operatorname{Re} f_2 - \operatorname{Im} f_{1|2} \operatorname{Im} f_2 = \operatorname{Re} f_{2|1} \operatorname{Re} f_1 - \operatorname{Im} f_{2|1} \operatorname{Im} f_1 \tag{20}$$

$$\operatorname{Re} f_{1|2} \operatorname{Im} f_2 + \operatorname{Re} f_2 \operatorname{Im} f_{1|2} = \operatorname{Re} f_1 \operatorname{Im} f_{2|1} + \operatorname{Re} f_{2|1} \operatorname{Im} f_1 \tag{21}$$

whence:

$$\operatorname{Re} f_{1|2} = \frac{\left(\operatorname{Re} f_{2|1} \operatorname{Re} f_1 - \operatorname{Im} f_{2|1} \operatorname{Im} f_1\right)\operatorname{Re} f_2 - \left(\operatorname{Re} f_1 \operatorname{Im} f_{2|1} + \operatorname{Re} f_{2|1} \operatorname{Im} f_1\right)\operatorname{Im} f_2}{\operatorname{Re}^2 f_2 - \operatorname{Im}^2 f_2} \tag{22}$$

$$\operatorname{Im} f_{1|2} = \frac{\left(\operatorname{Re} f_1 \operatorname{Im} f_{2|1} + \operatorname{Re} f_{2|1} \operatorname{Im} f_1\right)\operatorname{Re} f_2 + \left(\operatorname{Re} f_{2|1} \operatorname{Re} f_1 - \operatorname{Im} f_{2|1} \operatorname{Im} f_1\right)\operatorname{Im} f_2}{\operatorname{Re}^2 f_2 - \operatorname{Im}^2 f_2} \tag{23}$$

Thus, Eq. (23) (24) expresses the real and imaginary parts of the conditional density "amplitude" $f_{1|2}$ via the real and imaginary parts of the conditional and marginal density "amplitudes" $f_{2|1}$ f_1, and f_2. Here the departure from classical case is proportional to the degree of incompatibility (10) between the conditional densities Re $f_{1|2}$ and Re $f_{2|1}$.

It should be emphasized that for incompatible conditional processes the classical version of the Bayes' rule does not exist.

Since the physical meaning of probability is assigned to the real part of the density "amplitude" (but not to the square of its module as in quantum mechanics), there is no interference between different probabilistic branches of a stochastic process. This property allows one to simulate a complex version of the Fokker-Planck equation. Indeed, suppose that the components a and b of the complex joint density amplitude (12) satisfy the conditions which are sufficient for their representations as solutions to the corresponding Fokker-Planck equations, and, for the purpose of illustration, choose these equations in the following simple form:

$$\frac{\partial a}{\partial t} = D\left(\frac{\partial^2 a}{\partial x_1^2} + \frac{\partial^2 a}{\partial x_2^2}\right),\ \frac{\partial b}{\partial t} = D\left(\frac{\partial^2 b}{\partial x_1^2} + \frac{\partial^2 b}{\partial x_2^2}\right),\ D > 0 \tag{24}$$

Then, the complex "amplitude" (12) satisfies a complex version of the Fokker-Planck equation:

$$\frac{\partial f}{\partial t} = D\left(\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2}\right),\ f = a + ib \tag{25}$$

Assuming that

$$a = \delta(x_1, \xi_1, x, \xi),\ b = \delta(x_1, \eta_1, x_2, \eta_2)\ \ at\ \ t = 0 \tag{26}$$

where δ is the Dirac function,' one can verify that the corresponding incompatible stochastic processes from which the complex joint density "amplitude" (25) was derived, have the following conditional densities:

$$f_{1|2} = \frac{\exp\left[-\frac{(x_1-\xi_1)^2 + 2(x_2-\xi_2)^2}{4Dt}\right] + \exp\left[-\frac{(x_1-\eta_1)^2 + 2(x_2-\eta_2)^2}{4Dt}\right]}{\sqrt{4\pi Dt\left[\exp\frac{-2(x_2-\xi_2)^2}{4Dt} + \exp\frac{-2(x_2-\eta_2)^2}{4Dt}\right]}} \tag{27}$$

$$f_{2|1} = \frac{\exp\left[-\frac{2(x_1-\xi_1)^2+(x_2-\xi_2)^2}{4Dt}\right] + \exp\left[-\frac{2(x_1-\eta_1)^2+(x_2-\eta_2)^2}{4Dt}\right]}{\sqrt{4\pi Dt\left[\exp\frac{-2(x_1-\xi_1)^2}{4Dt} + \exp\frac{-2(x_1-\eta_1)^2}{4Dt}\right]}} \qquad (28)$$

Hence, a solution to the complex Fokker-Planck equation (26) subject to the initial conditions (27) can be simulated by two correlated (but incompatible) stochastic processes whose conditional densities are expressed by Eqs. (27) and (28).

In order to demonstrate a physical meaningfulness of the new mathematical formalism, define a joint entropy for two incompatible stochastic processes (27) and (28) as follows:

$$H = -\left|\int\int_{-\infty}^{\infty}(a\,\ell n\,a + ib\ell nb)dx_1dx_2\right| \qquad (29)$$

where $a(x_1, x_2)$ and $b(x_1x_2)$ are the components of the corresponding complex joint probability f defined by Eqs. (24) and (25).

We will show now that the entropy (29) represents a Lyapunov function for the evolution of the complex joint probability (25).

Indeed, denoting

$$H_1 = -\int\int_{-\infty}^{\infty} a\ell na \;\; dx_1dx_2, \quad H_2 = -\int\int_{-\infty}^{\infty} b\,\ell n\,b\,dx_1dx_2 \qquad (30)$$

one finds:

$$\frac{\partial H_1}{\partial t} = \geq 0, \qquad \frac{\partial H_2}{\partial t} \geq 0 \qquad (31)$$

since H_1 and H_2 *are the* Lyapunov functions for the solutions to Eqs. (24), respectively, which approach their maximum values at $t \to$ 00.

Therefore, the entropy (29) will approach its maxirnm value at $t \to \infty$ as a result of joint evolution of Eqs. (24), or of the evolution of their complex version (25).

It is important to emphasize that the entropy (29) consists of two components (see Eqs. (29) and (30)):

$$H = |H_1 + iH_2| = \sqrt{H_1^2 + H_2^2} \geq H_1 \tag{32}$$

Where H_1 and H_2 *are* defined by the real and imaginary parts of the joint probability $f = a + ib$, respectively. But the imaginary part b is proportional to the degree of incompatibility of the underlying correlated stochastic processes (27) and (28). Combining this result with the inequaaality in (32), one concludes, that incompatibility increases the entropy (and therefore, the information) of correlated stochastic processes.

Let us illustrate the situation described above by the following example. Consider a sensor data for the sea surface temperature T and the wind strength υ taken at the same point, Theoretically these data must be deterministic since they are solutions to the corresponding Navier-Stokes equations coupled with the equations of heat transfer. However, because of instability of these equations at super critical Reynolds numbers, actually the sensor data are random, and each time series can be treated as a sample of some underlying stochastic process. The Langevin-type representation of this process is very complex, and we will present it here in a symbolic form.

$$\dot{\upsilon} + \alpha(\upsilon, T) = \Gamma_1(\upsilon, T, t), \qquad \dot{T} + \beta(T, \upsilon) = \Gamma_2(T, \upsilon, t) \tag{33}$$

The first equation in (33) symbolizes the group of the Navier-Stokes equations written in the Reynolds form (prior to averaging) so that the Langevin-type force Γ_1 is represented by temperature-velocity fluctuations. The noise strength of this force can be expressed via the corresponding Reynolds stresses. As shown in [7], these stresses are proportional to the degree of instability of the original deterministic model.

The second equation in (33) stands for the group of heat transfer equation presented in a similar form.

Since Eqs. (33) are coupled, the conditions of their instability are coupled as well, and therefore, the Langevin-type forces Γ_1 and Γ_2 in general, must depend upon both the variables υ and T.

The corresponding symbolic Fokker-Planck equations will look as following:

$$\frac{\partial f_{\upsilon|T}}{\partial t} = \alpha \frac{\partial(\upsilon f_{\upsilon|T})}{\partial \upsilon} + \alpha R_\upsilon(\upsilon, T) \frac{\partial^2 f_{\upsilon|T}}{\partial \upsilon^2} \tag{34}$$

$$\frac{\partial f_{T|\upsilon}}{\partial t} = \beta \frac{\partial(T f_{T|\upsilon})}{\partial T} + \beta R_T(T, \upsilon) \frac{\partial^2 f_{T|\upsilon}}{\partial T^2} \tag{35}$$

Here $f_{\upsilon|T}$ and $f_{T|\upsilon}$ *are the* conditional probabilities of υ(given T) and T(given v), respectively; R_υ and R_T stand for the Reynolds stresses due to velocity and temperature fluctuations, while, (again, symbolically):

$$< \Gamma_1(t)\Gamma_1(t') > = R_\upsilon \delta(t - t') \tag{36}$$

$$< \Gamma_2(t)\Gamma_2(t') > = R_T \delta(t - t')$$

Although, in general, the probability evolution for Eqs. (33), is more sophisticated than those described by Eqs. (34) and (35), by introducing these equations we would like to emphasize here that it is nothing abnormal in the fact that the diffusion coefficients R_υ and R_T depend upon both variables.

Let us proceed with our example. Since the closed form solution to stochastic model of evolution for v and T cannot be obtained, we will assume that a family of v – time series at different T and a family of T-time series at different v are available.

Then one can represent these data (one way or another) in terms of probabilities. For simplicity, we will assume that they are expressed by Eqs. (27) and *(28),* where $v = x_1$ and $T = x_2$.

Then in order to develop a data-based stochastic model of the velocity-temperature evolution, one has to understand joint properties of these processes, and that is why a reconstruction of the joint probability is important. However, the processes (27) and (28) are incompatible (which can be verified by substitution in (10), and therefore, the joint probability should be sought in a complex form following from the system (18) with respect to a and b where $\rho_1 = \rho_{1|2}$, and $\rho_2 = \rho_{2|1}$ are given by Eqs. (27) and (28).

In our case, a and b are presented in the form (3) and (4) respectively (after replacements $D_1 = D_2 = D$) as solutions to the diffusion equations (24) subject to initial conditions (26). (For the purpose of illustration, we actually solved the inverse problem starting with Eqs. (24) and deriving Eqs. (27),(28)).

Thus the complex joint probability

$$f = a(v,T) + ib(v,T) \qquad (37)$$

found from sensor data can be exploited for studying two-dimensional evolution of T and v in the most general case.

This example describes incompatible stochastic processes appeared in nature A "man-made" processes of the same type are discussed in [6].

Generalization of the complex probability formalism introduced above to more than two variables has some restrictions: a complex joint density "amplitude" can be guaranteed for each pair of the conditional densities which are chosen from a set of all the conditional densities. However, the existence of joint density "amplitude" for more than two conditional densities imposes upon the latter the compatibility conditions similar to (1 O) even in the class of complex functions.

In conclusion, we will briefly summarize the results.

First] y, it has been identified a special class of incompatible stochastic processes which are coupled in such a way that their joint properties cannot be expressed by classical formalism. This new level of dynamical entanglement has been described by complex density "amplitudes."

Secondly, it has been demonstrated that incompatible stochastic processes can simulate new dynamical phenomena such as those described b y a complex Fokker-Planck equation. This new property enhances the capabilities of dynamical simulations for the purpose of information processing and computing, as well as for better modelling and identification of complex dynamical behaviors,

"The research described in this paper was performed by the Center for Space Microelectronics Technology, Jet Propulsion Laboratory, California Institute of Technology and was sponsored by the National Aeronautics and Space Administration, Office of Space Access and Technology."

The author thanks Dr. D. Bayard for interesting comments and discussions,

References

1. Gelfand, A. E., and Smith, A. F. M. (1990), Journal of the American Statistical Association, 85,398-409.

2. Schervish, M. J., and Carlin, B. P. (1990), Technical Report 492, Carnegie Mellon University, Dept. of Statistics.

3. Smith, A. F. M., and Gelfand, A. E. (1992), The American Statistician, 46,84-88.

4. Zak, M. Int. J. of Theoretical Physics, Vol 35. No. 2, 1996

5. Zak, M., Chaos, Solitons & Fractals, 1997 Vol. 8, No 5, pp. 793-804, 1997

6. Zak, M. Int. J. of Theoretical Physics, Vol. 33, No. 11, 1996

7. Zak, M. Int. J. of Theoretical Physics, Vol. 33, No. 11, 1994 pp. 2215-2280