A Simplified Theory of Coupled Oscillator Phase Control

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York and Liao have shown both theoretically and experimentally that one can control the phases of an array of coupled oscillators by tuning the oscillators on the perimeter of the array. They indicated the potential utility of this as a beam steering technique for phased array antennas. The theoretical demonstration was carried out using a complicated nonlinear formalism describing the behavior of injection locked oscillators. We have noticed that, in the "continuum limit" wherein the number of oscillators increases to infinity while their spacing is reduced to zero, the theory becomes reminiscent of electrostatics. In this presentation we develop this analogy and demonstrate its use in predicting the phase distribution for some given perimeter tuning configurate ions.

Outline

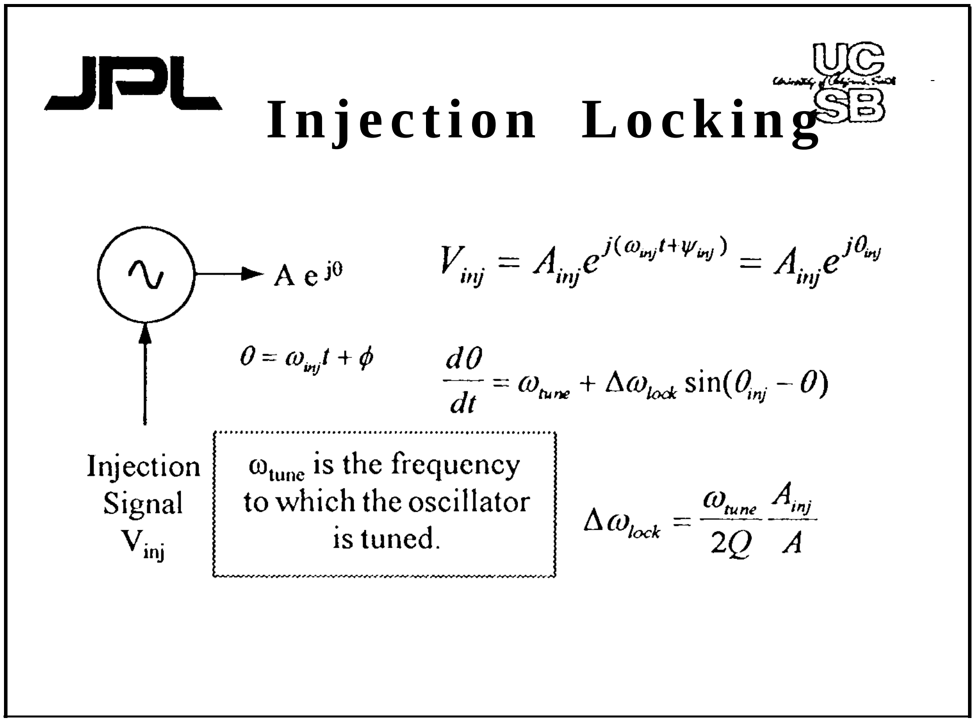
- Introduction
- Injection Locking
- Coupled Oscillator Theory
- ***The Continuum Limit***
- Some Example Solutions
- Concluding Remarks

We begin with a review of the work on beam steering via coupled oscillator control. We then consider the behavior of oscillators under injection locking and derive the simplified theory of this behavior in the array context. We then apply the theory to a few simple examples to illustrate the attainable effects and compare some of the results with those obtained using the full nonlinear theory.

Introduction

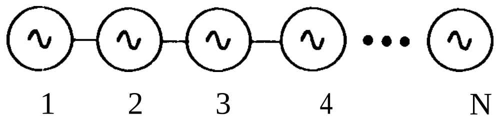
- Consider linear and planar arrays of coupled oscillators.
  - Achieve high radiated power through coherent spatial power combining.
  - Usually designed to produce constant aperture phase.
- Oscillators are injection locked to each other or to a master oscillator to produce coherent radiation.
- Oscillators do not necessarily oscillate at their tuning frequency.
- York, et. al. have shown that the phase of each oscillator is a function of the difference between the ***tuning frequency*** and the ***oscillation frequency.***

Our purpose in coupling oscillators together is to achieve high radiated power through the spatial power combining which results when the oscillators are injection locked to each other. York, et. al. have shown that the ensemble of injection locked oscillators oscillate at the average of the tuning frequencies of all the oscillators. Let's look at this in a bit more detail . . . .



Consider a single injection locked oscillator. We represent the signals as comples functions as indicated. In steady state, of course, the oscillator will oscillate at the injection frequency. The transient (time varying) behavior is governed by the indicated differential equation. Using this equation we can formulate the theory ofa set of coupled oscillators.

JPL Coupled Oscillators UCSB



$$\frac{d\theta_i}{dt} = \omega_{tune} - \frac{\omega_i}{2Q}\sum_{\substack{j=i-1 \\ j\neq i}}^{i+1} \varepsilon_{ij}\frac{A_j}{A_i}\sin(\Phi_{ij} + \theta_j - \theta_i)$$

In the continuum limit: $\frac{\partial \phi}{\partial t} = \omega_{tune}(x) - <\omega> + \Delta x\, \Delta\omega_{lock} \frac{\partial}{\partial x}\sin(\Delta x \frac{\partial \phi}{\partial x})$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{<\omega> - \omega_{tune}(x)}{(\Delta x)^2\, \Delta\omega_{lock}} = \rho(x)$$

$$<\omega> = \frac{1}{a}\int_0^a \omega_{tune}(x)\,dx$$

Poisson's Equation

Here we adapt the preceding differential equation to describe the behavior of a linear array of coupled oscillators with nearest neighbor coupling. Taking the contiuum limit of this description leads to Poisson's equation which, of course, also arises in electrostatics. The expression on the right of this equation is the analog of electric charge and is a function of the difference between the oscillation frequency of the ensemble and the tuning frequencies of the individual oscillators. Note that with proper tuning, one can establish the analog of any desired charge distribution. The solution of this inhomogeneous differential equation will be the phase distribution over the array. Of course, solution of this equation is quite straightforward using well known techniques.

One Dimensional Solutions

Using the Green's function: $\phi(x) = \phi_o + \frac{1}{2}\int_0^a G(x,x')\rho(x')dx'$

where $G(x,x') = -\frac{2}{a}\sum_{n=1}^{\infty}\frac{\cos\frac{n\pi x}{a}\cos\frac{n\pi x'}{a}}{\left(\frac{n\pi}{a}\right)}$, and $\rho(x) = \frac{}{(\Delta x)^2\Delta\omega_{lock}}\left[<\omega> - \omega_{tune}(x)\right]$

Ifthe end oscillators are detuned: $\omega(x) = 6.)_0 + \Delta\omega_1\delta(x') + \Delta\omega_2\delta(x'-a)$

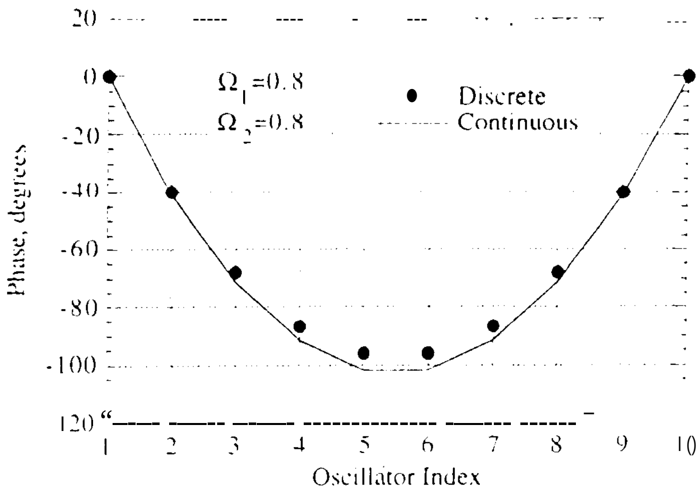
$$1\$(x) = \frac{\Omega_1+\Omega_2}{2a^2}x^2 - \frac{\Omega_1}{a}x \quad \text{where} \quad \Omega_i = \frac{a^2\Delta\omega_i}{(\Delta x)^2\Delta\omega_{lock}}$$

or, more symmetrically:

$$\phi(x) = \left(\frac{\Omega_2+\Omega_1}{2a^2}\right)\left(x-\frac{a}{2}\right)^2 + \left(\frac{\Omega_2-\Omega_1}{2a}\right)\left(x-\frac{a}{2}\right) - \left[\left(\frac{\Omega_2+\Omega_1}{2a^2}\right)\left(\frac{a}{2}\right)^2 - \left(\frac{\Omega_2-\Omega_1}{2a}\right)\left(\frac{a}{2}\right)\right]$$

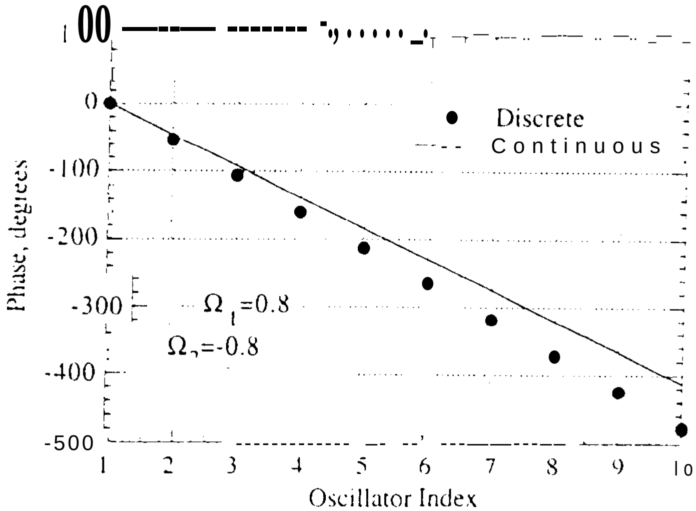
We can obtain the solution using the Green's function for Poisson's equation. We express this Green's function in the form of a series. Consider the "charge distribution" corresponding to detuning of the end oscillators in the linear array. The corresponding phase distribution is obtained by integration of the Green's function multiplied by the charge distribution and using the known summation of the resulting series. Not surprisingly, the solution is a parabola. Finally, we write the solution in a form which highlights the fact that the sum of the detunings controls the parabolic (quadratic) terms while the difference controls the linear terms which give rise to beam steering.

Symmetrical Detuning



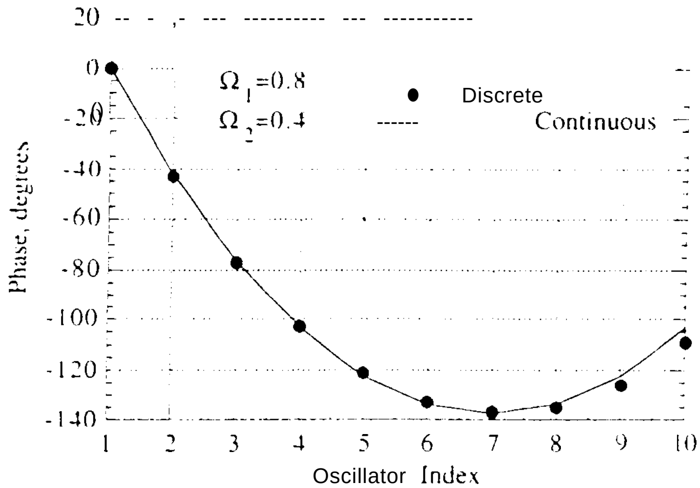
This plot show the result of symmetrical detuning of the end oscillators. The dots indicated the corresponding result obtained using the full nonlinear theory.

Antisymmetrical Detuning



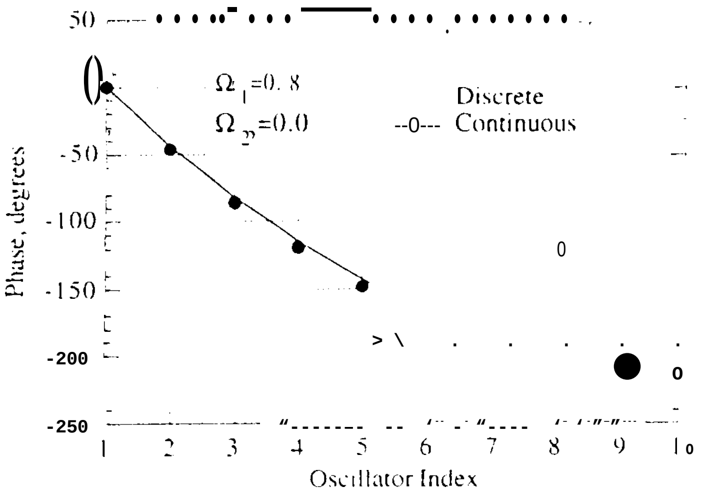
This plot show the result of antisymmetrical detuning of the end oscillators. The dots indicated the corresponding result obtained using the full nonlinear theory. This phase distribution results in beam steering.

Asymmetric Detuning



This is the result of an asymmetrical detuning configuration. Here again the dots represent the unapproximated theory.

One Sided Detuning



Here only the oscillator at one end of the array is detuned. Again the difference between the approximate and full theories is quite tolerable.

Two Dimensional Arrays

Poisson's equation in two dimensions:

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=\frac{<\omega>-\omega_{tune}(x,y)}{\Delta^2\,\Delta\omega_{lock}}=\rho(x,y)$$

The relevant Green's function is:

$$G(x,y;x'y')=\frac{4}{ab}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\frac{\cos\frac{\ell\pi x}{a}\cos\frac{\ell\pi x'}{a}\cos\frac{m\pi y}{b}\cos\frac{m\pi y'}{b}}{\varepsilon_{\ell 0}\varepsilon_{m0}\left[\left(\frac{\ell\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2\right]}$$

$$\varepsilon_{i0}=\begin{cases}2 & i=0\\ 1 & otherwise\end{cases}$$

We now move on to the analogous two dimensional case. Here again the solution is carried out using the Green's function for the two dimensional Poisson equation.

Two Dimensional Solutions

$$\omega_{tune}(x,y) = \omega_o + a\Delta\omega_{x1}\,\delta(x) + a\Delta\omega_{x2}\,\delta(x-a) + b\Delta\omega_{y1}\,\delta(y) + b\Delta\omega_{y2}\,\delta(y-b)$$

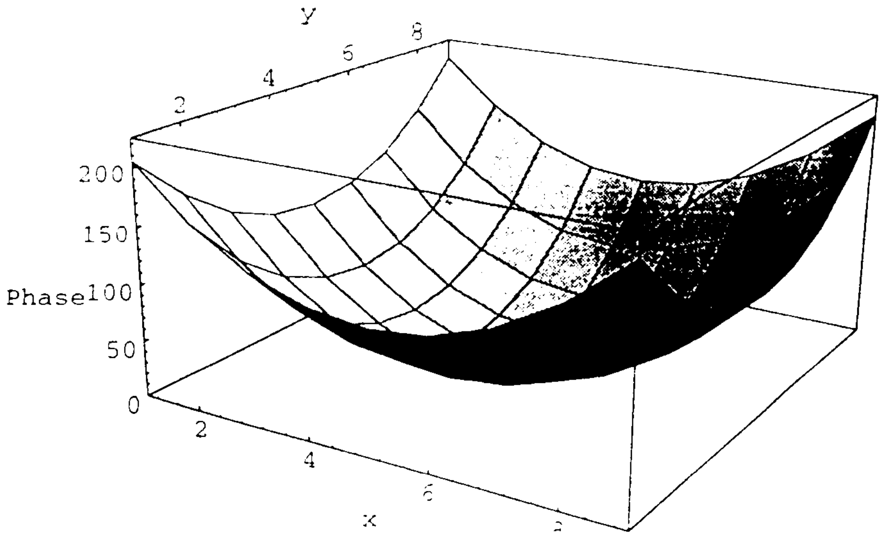
"Charge" distribution:
$$\rho(x,y) = \frac{(\Omega_{x1}+\Omega_{x2})}{a^2} - \frac{1}{a}\Omega_{x1}\delta(x) - \frac{1}{a}\Omega_{x2}\delta(x-a) + \frac{(\Omega_{y1}+\Omega_{y2})}{b^2} - \frac{1}{b}\Omega_{y1}\delta(y) - \frac{1}{b}\Omega_{y2}\delta(y-b)$$

Aperture phase:

$$\phi(x,y) = \phi_o - \left(\frac{\Omega_{2x}+\Omega_{1x}}{2a^2}\right)\left[\left(x-\frac{a}{2}\right)^2 - \frac{a^2}{12}\right] + \left(\frac{\Omega_{2x}-\Omega_{1x}}{2a}\right)\left(x-\frac{a}{2}\right) - \left(\frac{\Omega_{2y}+\Omega_{1y}}{2b^2}\right)\left[\left(x-\frac{b}{2}\right)^2 - \frac{b^2}{12}\right] + \left(\frac{\Omega_{2y}-\Omega_{1y}}{2b}\right)\left(x-\frac{b}{2}\right)$$

Considering again detuning of only the periferal oscillators, the charge distribution on the right side of the equation leads to a parabolic phase distribution. We arrange the result so as to highlight the fact that the difference in the detuning on opposite sides of the array control the beam steering while their sum controls beam "spoiling" through parabolic phase aberration.

Perimeter Detuning



This is the result of symmetrical detuning of the perimeter oscillators.

A Simplified Theory of Coupled Oscillator Array Phase Control

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Linear and planar arrays of coupled oscillators have been proposed as means of achieving high power rf sources through coherent spatial power combining. [1][2] In such applications, a uniform phase distribution over the aperture is desired. However, it has been shown that by detuning some of the oscillators away from the oscillation frequency of the ensemble of oscillators, one may achieve other useful aperture phase distributions. [3] Notable among these are linear phase distributions resulting in steering of the output rf bear-n away from the broadside direction. The theory describing the operation of such arrays of coupled oscillators is quite complicated since the phenomena involved are inherently nonlinear. This has made it difficult to develop an intuitive understanding of the impact of oscillator tuning on phase control and has thus impeded practical application. In this work a simplified theory is developed which facilitates intuitive understanding by establishing an analog of the phase control problem in terms of electrostatics.

We begin by reviewing the nonlinear equations describing the behavior of an array of loosely coupled oscillators. [2] The behavior of the phase of a single oscillator injection locked to an input signal,

$$V_{inj} = A_{inj} e^{j(\omega_{inj} t + \psi_{inj})} = A_{inj} e^{j\theta_{inj}}$$

can be described by the following differential equation.

$$\frac{\partial \theta}{\partial t} = \omega_o + \Delta\omega_{lock} \sin(\theta_{inj} - \theta)$$

where $\theta = \omega t + \phi$, ϕ is the phase of the oscillator oscillating at frequency, ω, and

$$\Delta\omega_{lock} = \frac{\omega_o}{2Q}\frac{A_{inj}}{A}$$

the locking bandwidth which is inversely proportional to the Q of the oscillator and A, the amplitude of the oscillation. Now, for an array of N coupled oscillators, the injection signals are just the outputs of the other oscillators and the phase of the i^{th} oscillator is described by a differential equation of the form,

$$\frac{\partial\theta_i}{\partial t} = \omega_o - \frac{\omega_i}{2Q}\sum_{j=1}^{N} \angle_{ij} \frac{A_j}{A_i} \sin(\Phi_{ij} + 9, - Cl_j)$$

and $\varepsilon_{ij}e^{j\Phi_{ij}}$ is the coupling between oscillators i and j. Limiting the coupling to nearest neighbors and taking the continuum limit as the number of oscillators increases to infinity and the spacing decreases to zero (i becomes a continuous variable, x), results in,

$$\frac{\partial\phi}{\partial t} = \omega(x) - <\omega> + \Delta\omega_m \frac{\partial}{\partial x}\sin\left(\frac{\partial\phi}{\partial x}\right)$$

where $\Delta\omega_m$ is the mutual locking bandwidth of the coupled oscillators and $<\omega>$ is the average of the oscillator tuning frequencies, o(x). In steady state with small phase differences between neighboring oscillators, one has,

$$\frac{\partial^2\phi}{13x^2} = \frac{<\omega> - (1)(x)}{\Delta\omega_m} = p(x)$$

which is Poisson's equation of electrostatics! Similarly for a two dimensional array one obtains a two dimensional Poisson equation. From this point, all of the familiar results of electrostatics apply if one merely identifies the oscillator tuning with charge density and the phase distribution with electrostatic potential.

For example, suppose that we detune the oscillators at each end of a linear array in opposite direction with respect to the average tuning frequency with the intention of steering the beam as described by Liao and York. [3] This can be represented as two delta function charge densities of opposite sign one at each end of the aperture. The solution for the phase distribution is merely a linear function as shown in Figure 1. yielding the desired steering of the beam. This linear solution may, of course, be recognized as the potential in a parallel plate capacitor. For comparison, the dots in Figure 1 represent the solution of the full nonlinear equations with no approximation.

Note that if the two delta functions have the same sign, the average of the tuning frequencies is changed resulting in a constant charge distribution in addition to the deltas. This constant term yields a quadratic solution for the phase distribution as shown in Figure 2. Of course, various ratios of delta function amplitudes yield corresponding combinations of linear and quadratic solutions such as the one indicated in Figure 3. Similar results obtain for two dimensional arrays wherein, for example, various detunings of the oscillators on the perimeter of the array yield phase distributions which are solutions of the two dimensional Poisson equation with delta functions and constants as sources. Such a phase distribution is illustrated in Figure 4. This resulted from detuning of all the perimeter oscillators by the same amount.

Finally, it is noted that this simplified theory makes clear the fact that any desired slowly varying phase distribution can be realized if one is willing to detune all of the oscillators. The appropriate tuning can be ascertained by substituting the desired phase distribution into Poisson's equation and determining the resulting charge distribution.

References

1. J. W. Mink, "Quasi-optical power combining of solid-state millimeter-wave sources," IEEE Trans. Microwave Theory Tech., vol. MTT-34, pp. 273-279, Feb. 1986.
2. R. A. York, "Nonlinear Analysis of Phase Relationships in Quasi-Optical Oscillator Array s," IEEE Trans. . Microwave Theory Tech., vol. M'I"f-41, pp. 1799-1809, Oct. 1993. [See also references therein.]
3. P. Liao and R. A. York, "A New Phase-Shifterless Beam-Scanning Technique Using Arrays of Coupled Oscillators," IEEE Trans. . Microwave Theory Tech., vol. MTT-41, pp. 1810-1815, Oct. 1993.

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Figure 1. Equal and opposite detuning of the end oscillators.

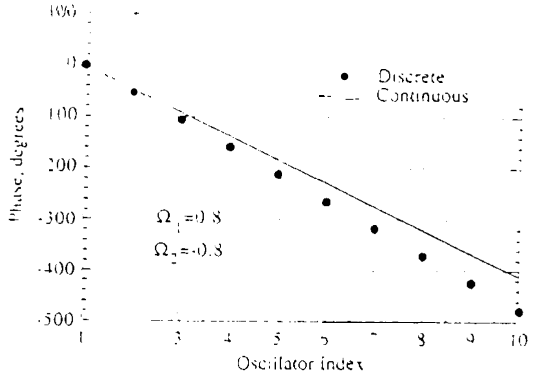


Figure 2. Equal detuning of the end oscillators.

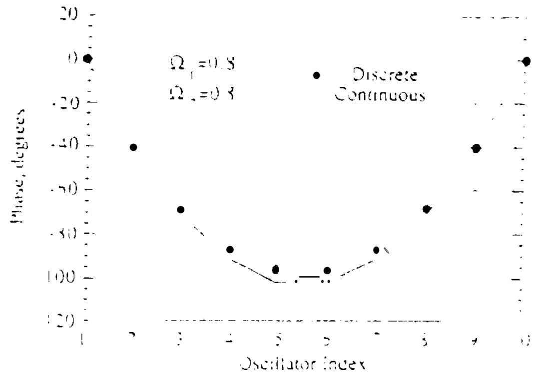


Figure 3. Unequal detuning of end oscillators.

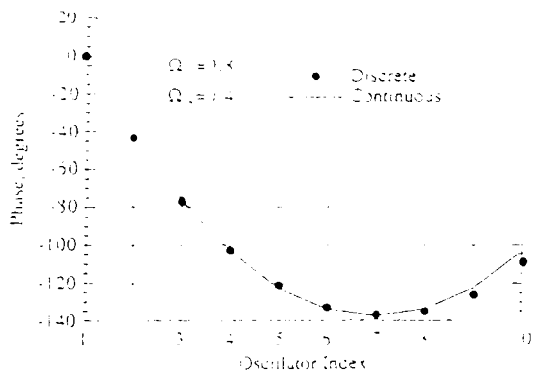
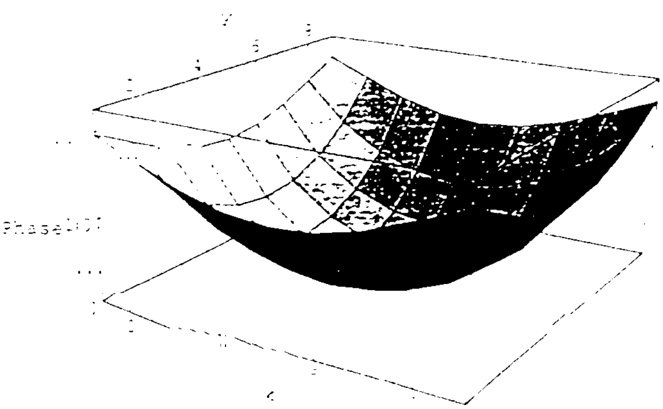


Figure 4. Equal detuning of perimeter oscillators.



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Section 336

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PHASE CONTROL

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Author(s): R. J. POGORZELSKI AND R. A. YORK

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