Thomson's Lamp is Dysfunctional

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ABSTRACT

James Thomson envisaged a lamp which would be turned on for 1 minute, off for 1/2 minute, on for 1/4 minute, etc. *ad infinitum.* He asked whether the lamp would be on or off at the end of 2 minutes. Use of “internal set theory” (a version of nonstandard analysis), developed by Edward Nelson, shows Thomson’s lamp is chimerical; its copy within set theory yields a contradiction. The demonstration extends to placing restrictions on other “infinite tasks”: Zeno’s paradoxes of motion; Kant’s First Antinomy; and Malament-Hogarth spacetimes in General Relativity. Critique of infinite tasks yields an analysis of motion and space & time; at some scale, motion would appear staccato and the latter pair would appear granular. The critique also shows necessary existence of some degree of “physical law”. The suitability of internal set theory for analyzing phenomena is examined, using a paper by Alper & Bridger (1 997) to frame the discussion.

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1. THOMSON'S LAMP

Zeno of Elea (c.490-30 B. C.) formulated about 40 *epicheirêmata* (attacks) for the purpose of disparaging motion and plurality and in support of the monistic worldview of Parmenides of Elea (born c.515 B.C.). Zeno's writings have not survived, but Aristotle (384-22 B. C.) has preserved significant fragments of the younger Eleatic's thought and, in the course of his commentary, states what it would mean for one to claim to undertake an "infinite task".

Zeno's paradox of motion known as "The Dichotomy" argues that before a moving object could reach a goal it would first have to traverse half the remaining distance, then half again, etc. *ad infinitum.* In his *Physics* (translation of Waterfield, 1996, "263'4") Aristotle comments: "We should make the same response to anyone who uses Zeno's argument to ask whether it is always necessary to traverse half the distance first, and points out that there are infinitely many half-distances and that it is impossible to traverse infinitely many distances; or then there are others who put the same argument another way and maintain that, as one moves over a half-distance, one has to count it before completing it, and has to do so for each half as it happens, and so traversing the whole distance turns out to involve having counted an infinite number which is admittedly impossible." For Aristotle, then, the infinite task is based on the process of counting.

But Aristotle does not concur with Zeno that motion is impossible; rather, the infinite task posed by The Dichotomy is an artificial construct, a purely mathematical objection which can be ignored. "So the reply we have to make to the question whether it is possible to traverse infinitely many parts (whether these are parts of time or of distance) is that there is a sense in which it is possible and a sense in which it is not. If they exist actually, it is impossible, but if they exist potentially, it is possible. I mean, anyone in continuous motion has coincidentally traversed infinitely many distances, but he has not done so in an unqualified sense; it is a coincidental property of a line that it contains infinitely many halves, but it is not essential to what it is to be a line." ("263'3") For him, although the real line is potentially divisible any number of times, actual division is not a necessary concomitant with motion: there is no "task" involved, only analysis. Aristotle holds that The Dichotomy does not mandate the completion of an infinite task.

Leo Groarke (1 982) also does not believe an infinite task can be completed. However, he differs from Aristotle through the belief that The Dichotomy does mandate such completion. He states "The Principle of Sequential Acts (PSA)": "The performance of a sequence of acts does not complete a particular task unless it is completed by the performance of one of the acts in the sequence. " Groarke concludes: "In the present context, we need not consider PSA in detail. It is not difficult to make a case for it, and it straightforwardly follows that no infinite sequence of acts can be completed. . . . Until philosophers have a better answer to The Dichotomy paradox, it is premature to contemplate the completion of any such sequence."

Richard Sorabji (1983) is sanguine about the accomplishment of an infinite task. “The solution I would favor simply denies Aristotle’s claim that it is impossible to traverse an infinity. In denying this, I must also depart from Aristotle’s conception of infinity as something which is always incomplete, and which is to be understood in terms of the possibility of adding to a *finite* collection of divisions. Speaking in an idealized way, we can view ourselves as traversing a *complete* infinity of sub-distances every time we move.” Sorabji not only contradicts Aristotle, he rejects Groarke’s “Principle of Sequential Acts” while distinguishing between the members of a sequence and the sequence as a whole. When one considers the whole sequence, there is, for Sorabji, no paradox.

James Thomson (1 954-55) has conceived a way of expressing an infinite task so, unlike Zeno’s Dichotomy, it would almost certainly be acknowledged as compulsory. Thomson’s lamp is turned alternately on and off for lengths of time which converge geometrically, e.g., 1 minute on, 1/2 minute off, 1/4 minute on, etc. *ad infinitum.* At a time equal to the limit of the geometric series (2 minutes for the example), is the lamp on or off? Paul Benacerraf (1962) has characterized Thomson and a few others who have devised such puzzles as “the modern Eleatics”. In the present paper, “Eleaticism” will be used to denote the contributions of both ancient and modern Eleatics.

Thomson's infinite task appears to be mandatory, but there is motive to find grounds to reject the concept because either luminal outcome is paradoxical. We shall not follow the route of investigating questions of engineering feasibility for constructing the lamp; such endeavors are notoriously short-sighted. (We will, in the final section, touch on quantum-theoretic constraints on observations and measurements of phenomena.) Instead, we will show there is a logical problem with Thomson's formulation, a problem which can be uncovered through mirroring the operation of the device within set theory and demonstrating a resultant contradiction.

Demonstration that Thomson's lamp is dysfunctional is the lead topic for the present work, but several other problems lying within Eleaticism are also addressed. The reason for this comprehensive approach has its root in the multiplicity of proposed explanations attached, historically, to each problem within the domain of Eleaticism. This variegated nature of the class of explanations is itself a problem but one with an obvious methodological response: add the criterion of degree-of -comprehensiveness in judging any proposed explanation. An explanation will be "comprehensive" insofar as it is applicable to a wide range of problems within Eleaticism, and it possesses the potential for extension to matters beyond that domain.

The principle underlying the present approach, the "critical mensuration thesis", is: every phenomenon can be completely described through the use of real numbers, but not all real numbers can be used for describing phenomena. The first clause, the "mensuration thesis", in the statement of the greater thesis, rests upon the success of experimental science. The second clause must be argued, and this is carried out through the medium of internal set theory, to be introduced in the next section. ("Critical" is used in the name of the greater thesis because, as will be seen, the "critical mode" of internal set theory, as opposed to the "theoretical mode", is used.)

The approach in the present work is comprehensive through addressing a half dozen or so problems of varied textures within Eleaticism, and its extension demonstrates the staccato nature of motion, the granularity of space & time (at some scale), and the necessary existence of some degree of "physical law".

2. INTERNAL SET THEORY

Since the time of Aristotle, logic has constituted an important branch of philosophical investigation. It is treated both as an *organon* and as a part of philosophy proper. In the late nineteenth century, Georg Cantor's (1845-1918) creation of set theory produced a second symbolically-based adjunct to philosophy. (Modern logic, at least, can be fairly described as "symbolically based".) Some, such as the logician Kurt Gödel (1906-78), have hypostatized sets, producing a latter day Theory of Forms (appropriately, this view of set theory is called "Platonism"). Such ontological claims place set theory within the domain of philosophy (as well as its more robust roles within mathematics and the philosophy of mathematics). Here, however, set theory will function solely as an *organon* with which to apply a critique, in the next section, of Thomson's lamp and of other infinite tasks.

Zermelo-Fraenkel set theory is the most common formulation now in use by mathematicians. The theory can be based upon nine axioms, which specify how “set” is to be used, e.g., see Cohen, 1966. (Strictly speaking, there are an infinite number of axioms because one is a “schema”, whose parameterization entails an infinite number of axioms.) “Zermelo-Fraenkel set theory” will be denoted by “ZFC”; “C” denotes one of the axioms, the “axiom of choice”. A perfectly good version, “ZF”, of set theory can be had by omitting “C”. The “axiom of extensionality” of ZFC is required subsequently:

(1) $(\forall x)\ (\forall y)\ (\forall z\ (z \in x \leftrightarrow z \in y) \rightarrow x = y)$.

This axiom says that a set is determined by its members.

Edward Nelson (1977) adjoins three axioms to ZFC, creating “internal set theory” (1ST). It can be shown that 1ST is a consistent and conservative extension of ZFC: “consistent” because if ZFC is consistent (i.e., one cannot prove both A and $\neg$A), then so is the augmented system; “conservative” because, informally speaking, no new theorems within ZFC are made possible by the new axioms. (In language to be explained below, “every internal statement which can be proved in 1ST can be proved in ZFC”.) Internal set theory lies within the tradition of “nonstandard analysis” introduced by Abraham Robinson (1974).

Nelson’s formulation begins with introduction of a new predicate, “standard”, a unary relation which applies to sets (all objects, except logical constants, within IST are sets, i.e., there are no logical atoms). Objects which are not standard are “nonstandard”. A formula within 1ST is called “internal” if it does not utilize the predicate “standard” (all formulas of ZFC are internal). Formulas which are not internal are called “external”, and the simplest external formula is “x is standard”.

It is crucial to recognize that the predicate “standard” is not analyzable into simpler parts in terms of other mathematical notions; it is not defined. Rather, it is characterized through three axioms (given below). The binary relation “$\in$” is also fundamental and not defined while the binary relation “<" can be defined (in terms of “$\in$“). The consequence of this fact is that external formulas cannot be used to define sets. The axiom of specification in ZFC does allow one to define sets with (internal) formulas. Let P be an internal formula, then,

(2) $y = \{ = x \mid P(z)\}$

yields a set y as a subset of x. But, the axioms of ZFC have, so to speak, no knowledge of the predicate “standard”, and it cannot be expressed in terms which would allow it to be employed in (2). In other words, P cannot be used in (2) if P is an external formula. Nelson calls such attempts “illegal set formation”.

The three axioms which are added to those of ZFC in order to establish 1ST are the “transfer principle”, the “principle of idealization”, and the “principle of standardization”.

The first, transfer, is

(3) (v” x) P(x)+ (∀ x) P(x),

where “V$^{s[}$” means ‘(for every standard x“, and P is an internal formula. This principle allows us to transfer assertions of classical mathematics (ZFC) to 1ST. Of course, the reverse implication also holds (because “∀ x“ includes “V^{s}’ x“). Forming the contrapositive of (3) yields,

(4) (3x) P (x)+ (3^{s}’ x) P (x).

If there is only one x such that P (x) holds, then (4) allows us to conclude that x must be a standard set. Nelson paraphrases this conclusion: “every specific object of conventional mathematics is a standard set”. This characterization suggests that nonstandard objects are elusive; since they cannot be obtained through the devices of conventional mathematics, they are, in this sense, ineffable.

So far, no nonstandard objects have been introduced. The principle of idealization provides a remedy for that lack.

(5) $(\forall^{st\,fin} z)(\exists x)(\forall y \in z)\ R\ (x,y) \leftrightarrow (\exists x)(\forall^{st} y)\ R\ (x,y).$

Here, R is an internal formula and “V^{st}“n” reads “for every standard and finite”. Letting R be the binary relation “less than” (“<”), then (5) can be used to show the existence of “infinitesimal” real numbers. For any standard and finite set z of positive real numbers, it is trivial to observe that there is a real number x that has the properties of being positive and less than all $y \in z$. Such x will, in general, vary with the choice of z; no matter, one can, by (5), assert the existence of an (“ideal”) element which is greater than O but less then all standard, positive, real numbers. Such a nonstandard real number seems very small, from the way it was obtained, and is called an “infinitesimal”. (The number O is also included, by a definition, among the infinitesimals.) The inverse, also nonstandard, of a nonzero infinitesimal must be very large and is called an “unlimited” real number (but, being a real number it is, perforce, a finite object). It is important to note that no new objects have been added to the real-number system; infinitesimals and unlimited real numbers are objects from ZFC that are seen in a new perspective.

The third axiom, standardization, allows one to satisfy, to a certain degree, the craving to form subsets using external formulas,

(6) $(\forall^{st} A)(\exists^{st} B)(\forall^{st} x)(x \in B \leftrightarrow x \in A \wedge P(x))$,

where P is a formula which can be internal or external. The part "$x \in A \wedge P(x)$" of (6) is a subset-forming technique: all x belonging to A that have the property P. Nelson supplies a caution to accompany standardization: "When a standard set is defined by the standardization principle, the criterion for set membership applies only to standard elements. "

With the axiomatic basis of IST in place, some basic theorems are listed. For their proofs, Nelson (1977) should be consulted. (This work is clearly written and furnishes the best introduction to 1ST, touching upon several areas of mathematics, with examples.)

1. Every element of a set is standard if and only if the set is standard and finite.

2. A corollary is that every infinite set contains a nonstandard element. Conjoined here are two objects, an infinite set and a nonstandard element, each of which is remote from experience; they are objects of theory, not of the world.

3. There isafinite set Fwhichcontains all standard elements. This isa surprising theorem and can be proved with relative ease, using the principle of idealization. It cannot be concluded that the “set” of all standard objects is finite because it would be a case of illegal set formation to attempt to form this collection.

In addressing Thomson’s lamp, and allied subjects, the objects under consideration will be real numbers. Within 1ST, nonstandard real numbers can be grouped into three classes, two of which, infinitesimals and unlimited real numbers, have already been encountered. The third, “mixed nonstandard real numbers”, is composed of reals which consist of a standard real number plus or minus a (nonzero) infinitesimal. The real line can be pictured as a collection of standard real numbers, each of which is surrounded, “infinitely closely”, by an enclave of mixed nonstandard reals (the number O is surrounded by infinitesimals), and both extremities of the line are populated exclusively by unlimited real numbers.

3. DYSFUNCTIONALITY OF THOMSON’S LAMP

Most of the work necessary for proving dysfunctionality is done; the result is encoded in 1ST, and it only remains to trace the outline.

Since we are eschewing engineering constraints, it is not difficult to suppose that a counting device has been added to the switching apparatus of the lamp. (This is consistent with Aristotle's conception, above, of the essence of an infinite task.) By "counting" we mean to indicate an activity as concrete as the contemplated pulses of light from the lamp: not, for example, an abstract process such as finite or transfinite mathematical induction. For specificity, assume that counting is instantiated through the "stroke method": one stroke is physically recorded each time the lamp changes state. Thus, after 1 and 7/8 minutes (on, off, on, off) the counter (a piece of paper, an abacus, an electronic device, etc.) reads, in stroke-system notation,

(7) | | | |.

Of course, any other notation which provides a concrete link to the number 1 (one) would suffice, i.e.,

(8) 4.

What we are trying to avoid is abstract or indefinite representations such as "n". The mensuration thesis for Thomson's lamp identifies the purported functioning of the lamp with the counting process described above.

In order to accomplish the infinite task seemingly mandated by the modified Thomson's lamp, it is certainly necessary to count to some unlimited (and nonstandard) natural number n. But if this has been done, then n has been represented as an object of “conventional mathematics”. Hence, n must also be a standard natural number. The contradiction completes the *reductio ad absurdum* and shows that the lamp cannot even complete a certain finite task; *a fortiori,* the infinite task (counting all of the natural numbers) cannot be completed. Thomson’s lamp is dysfunctional. The “critical” aspect of the critical mensuration thesis is exemplified for Thomson’s lamp by the above *reductio*.

This resolution of the infinite task posed by Thomson’s lamp accords with Aristotle’s judgment, “[the task] involve[s] having counted an infinite number which is admittedly impossible”.

Zeno’s Dichotomy illustrates three facets of an infinite task: an ordinal facet, a labeling one, and a facet dependent upon indiscernibility. First, there is an ordinal infinite task which is similar to Thomson’s lamp: running through the “Checkpoint sequence”, S_n= 1 - 1/2”, n = O, 1, 2, ... where for the mensuration thesis we adopt Aristotle’s view that, if accepted, this task consists of counting. Aristotle was correct; it is better not to accept the task. Like Thomson’s lamp, the task would be logically flawed with respect to its formulation, as Groarke has supposed. Sorabji would not accept the identification of the infinite task with counting but would move through difficulties aided by more sophisticated mathematical concepts. Since we have not resolved (except to the satisfaction of Parmenides and Zeno) The Dichotomy through this approach, it is necessary to go deeper into the structure of the paradox.

The second facet comes from recognition that there is a problem with the use of any nonstandard real number (and not just an unlimited natural number n) and, in particular, S_n, as a measurement label. (See below.) Assuming this to be the case, we could not possibly verify the fact of the object being located at a point labeled by nonstandard S_n (and since there is an infinite number of S_n, some must be nonstandard), and Zeno's argument would be moot. Physical paradox cannot be derived from nonverifiable behavior; one just rejects the theoretical scenario, here, the premise of The Dichotomy, which is supposedly causing the difficulty. This approach was used by McLaughlin & Miller (1992) to resolve The Dichotomy. The mensuration thesis identifies The Dichotomy with the elements of the Checkpoint sequence, and the critical use of 1ST reduces the phenomenal realm to benign proportions.

It is worth a brief excursion to note why one cannot resolve The Dichotomy within standard analysis by application of the theory of limits. In view of the prevalence of topological concepts of convergence during most of the twentieth century, it is somewhat surprising to see the frequency with which the theory of limits is proposed as a resolution of The Dichotomy. The Checkpoint sequence of Zeno will converge for some topologies (e.g., the Euclidean) and not for others (e.g., the right half-open interval: see Steen & Seebach, 1978, for definition). Hence, one must make an extra-mathematical argument as to why the Euclidean topology (the usual choice) should be selected. The course of any such argument is not obvious. Consider one domain of opinion, modern physics. Kip Thorne (1 994) imagines a microscope examining "space" with progressively greater degrees of magnification. "At all the early, 'large' scales, space would look completely smooth, with a very definite (but tiny) amount of curvature. As the microscope zoom nears, then passes 10-32centimeter, however, one would see space begin to writhe, ever so slightly at first, and then more and more strongly until, when a region just 10^{-33} centimeter in size fills the supermicroscope's entire eyepiece, space has become a froth of probabilistic quantum foam." The endurance of the Euclidean topology for mathematical analysis of The Dichotomy might be explained by tradition and mathematical conventions like the identification of the base-2 expression "0.1 111. . ." with "l".

Since the epistemological status of nonstandard real numbers is at the center of the argument of the present work, the subject will be investigated now. It not only ties off The Dichotomy but also allows a new perspective on Thomson's lamp. The argument which established the dysfunctionality of Thomson's lamp has already shown that one cannot know unlimited natural numbers, in the sense of relating them concretely to 1 ("counting from 1 "). They are not "accessible".

Consider the three types of nonstandard real numbers: unlimited, infinitesimal, and mixed. If one claimed to be in possession of significant specific details of a certain unlimited real number, call it "r", then this knowledge would be expected to include the ability to count to [r] (where [x] denotes the largest integer less than x, i.e., the "integral part" of x). But this cannot be done, as previously demonstrated. Hence, r is not accessible. Infinitesimals are the inverses of unlimited real numbers, so they, too, are inaccessible. A mixed real number is composed of a standard real number plus or minus an infinitesimal; it is inaccessible.

To return to The Dichotomy ("second facet"), recall again that every infinite set contains a nonstandard object. Hence, at least one of Zeno's Checkpoint-sequence members is nonstandard, and it can represent no matter of fact, even in principle, no matter how capable observing systems might become. In fact, all Checkpoint-sequence members within any infinitesimal distance from 1 must be nonstandard objects, and hence inaccessible.

With regard to Thomson's lamp, suppose that we had no knowledge of the ordinal argument which established its dysfunctionality, and (counterfactually) the lamp was emitting its ever shortening pulses of light. Then, within a time infinitesimally close to 2 minutes, the lamp's behavior is not observable (even if, in some way, it were to be emitting pulses of light), This is reminiscent of certain quantum-mechanical results which will be discussed in the last section. The critical mensuration thesis has erased the pulses of light during an infinitesimal segment of time prior to the temporal terminus.

The third facet of The Dichotomy is a spatial analog of the above-cited behavior of Thomson's lamp. For The Dichotomy itself, the infinitesimal region near the spatial terminus is filled with Checkpoints which are indiscernible, one from the other.

Note that in the above analysis of The Dichotomy (and the lamp), there has never been a question as to whether or not space (or time) is somehow “composed” of points, standard or nonstandard. We consistently adopt the perspective of an observer who is measuring phenomena and have shown that nonstandard numbers are not available as measurement labels for those phenomena, The geometrical term “point” is used as a descriptive aid.

If, in Kant’s First Antinomy (Kemp Smith, 1929), we were to contemplate the origin of the universe at an infinitely remote time, then a sequence of calendric points could be constructed to render the temporal unfolding as, essentially, “version 2“ of The Dichotomy (McLaughlin & Miller, 1992). (In that version, before an object reaches 1/2, it must first reach 1/4, etc. *ad infinitum, so* it can never start to move.) So, as with The Dichotomy, the unfolding can proceed if we do not insist upon information transfer during the process from “too many” points of time. That is, the universe could evolve from an infinitely remote past, but, like Faust, we encounter problems if we seek to learn too much about the process.

Max Black (1 954) has postulated, in response to Aristotle, a trajectory which does have an infinite number of distinguished points (unlike points passively residing in [0, l]): a ball which is thrown, hits the ground, and repeatedly bounces exhibits (ideally) an infinite number of vertical maxima. This is clearly the same behavior as the first version of The Dichotomy and can be similarly understood. Eventually, the diminishing maxima achieve only infinitesimal altitudes and become, for this reason, purely theoretical entities with no ability to “mark” the trajectory. If the coefficient of restitution is 1, so that the maxima do not decrease, an ordinal counting argument shows that only a limited-finite number of distinguished points can ever be observed.

John Earman (1995) has suggested that the so-called Malament-Hogarth class of spacetimes, in the context of General Relativity, may be agents for the accomplishment of infinite tasks (such as calculating and printing all the digits of π). He makes use of their peculiar properties and the relativistic concept of “proper time”; an observer, using only a finite amount of his or her proper time is nevertheless able to view the calculator, who operates for an infinite amount of his or her proper time. Though ingenious of design, like Thomson’s lamp the concept is wrecked by the same logical rock. The problem centers, of course, on the inability of the calculator to complete the infinite task, regardless of the particulars of the relationship between observer and calculator. Although the parallel is not exact, the Malament-Hogarth scenario can be considered to be a particular type of manifestation of Kant’s First Antinomy.

4. STACCATO MOTION AND THE GRANULARITY OF SPACE & TIME

A certain desiccation invests observed motion as characterized in the preceding section. Only standard real numbers are candidates to be measurement labels with respect to phenomena, and a countably infinite number of intervals of infinitesimal length, containing only nonstandard real numbers, have no role to play. This notion can be made more precise,

Let an object be moving in [0, 1], and consider the most complete physical description which could be given of these phenomena. By the mensuration thesis there is a (finite) set A $\subseteq$ S, where S $\subseteq$ F (see Section 2 for F) contains all standard real numbers in [0, 1], of real numbers which supports this description. Each element of A is a standard real number since nonstandard numbers cannot be used as measurement labels. The set A must be of limited-finite cardinality because the distance between any two standard real numbers cannot be infinitesimal. Not every standard element of S can belong to A or else, through the axiom of extensionality, we would have supposedly created the set of all standard real numbers in [0, 1]. Within this supposed set the real number $(r_j + r_{j+1})/2$, midway between two adjacent numbers, r_j and r_{j+1}, must be standard, contradicting the presumed adjacency of r, and r_{j+1}. The case of one motion through [0, 1] can be extended to all observable motions through [0, 1} and, more generally, to all phenomena, and still A (and similar sets for cases with other phenomena) remains of limited-finite cardinality.

There are four implications from these properties of A. First, motion appears to proceed in jumps; it is staccato as observed. Second, “physical law” (an extra-mathematical concept), of some sort, is necessary in order to extract the reduced set A from S. Third, the “granularity” of space (and time, which is always measured by observing the functioning of some natural or artificial object, i.e., a series of phenomena) isaconsequence of therestricted cardinality of A. Last, the fact that the elements of A are separated, each from the others, by noninfinitesimal distances makes it possible to employ theories of error for measurements.

The theories of the Greek atomists and, hence, the prospect of the granularity of space (and time) may have motivated Zeno's paradox of The Stadium, which exposes problems of calculation in such an environment. (A modern analysis of granularity is given by Forrest (1995)). Hermann Weyl (1946), with the aid of a simple figure, produces the "tiling argument", showing that the length of the hypotenuse of a right triangle would equal the sum of the lengths of the two sides, in a space with square grains. Van Bendegem (1987) restores the usual Euclidean result by taking into account the thickness of the elements of the triangle. Note that if the tiles have infinitesimal dimensions, then Van Bendegem's calculations need not be entered into because the tiles would be purely theoretical objects, and there is no anomaly to be explained once one leaves observable domains.

The concept of a granular space, derived from A, is easily formalized with the construction of a topological space having the "identification topology" (Gemignani, 1972). Indiscernible points are collected into equivalence classes ("grains"), which form the points of the new space. One is appealing to Leibniz's principle of the "identity of indiscernible" in creating the equivalence classes. These identification topologies form a class; there is not a unique one which can be prescribed from purely mathematical considerations. "Physics" is required in order to specify the actual identification topology which holds for space & time. The construction extends to higher-dimensional spaces in a straightforward way, e.g., through the employment of a product topology.

Zeno's paradox of The Arrow has been resolved through, essentially, appealing to the granularity of space (McLaughlin & Miller, 1992) because no Zenonian commentary about motion within the granules could ever be verified. But the inaccessibility of nonstandard real numbers can also be utilized to facilitate theoretical explanations. In this mode, as early as the third century B. C., Chrysippus may have suggested motion by infinitesimal increments (White, 1982), a nonparadoxical scenario but not verifiable. Giovanno Benedetti in the sixteenth century also considers the use of infinitesimals in this context (Cajori, 191 5). Michael White (1 982, 1992) has conducted an imaginative synthesis of ancient philosophy with modern logic and proposes the use of nonstandard analysis (the version created by Abraham Robinson) to resolve The Arrow. McLaughlin & Miller (1 992) propose a similar theory of motion.

The results of this section say nothing about what physical objects might populate the lacunae represented by the grains or what laws of motion actually control staccato motion or if these laws are invariant with time or if they are deterministic or probabilistic. For this trace level of influence, we might speak of "physical intervention" in place of "physical law"; it is only the historical success of scientific explanation which recommends the second term. The term "granularity" has been chosen to avoid connotations of matter which adhere to "atomic". Also, the scale of the granularity is not known; it might well be so fine that its presence will never be detected through observations.

Mathematical constraints, of a very general nature, on the physical world are not uncommonly proposed. See, for example, the reasoning of Lindsay & Margenau (1 957) as to how second-order differential equations might still be applicable to a thoroughly acausal universe or Richard Feynman's delightful pastiche (Van Ness, 1969) of the law of the conservation of energy, revealing its basis in accountancy.

5. NOTES TOWARD A PHILOSOPHICAL ESTIMATE OF INTERNAL SET THEORY

The arguments that disable Thomson's lamp and provide constraints on other infinite tasks stand on their own merits, but they are related to more general approaches, mathematical and otherwise, to Eleaticism. The following classification displays one way to taxonomize strategies for dealing with infinite tasks.

A. argue for impossibility of task
    1. on logical grounds
    1. on engineering grounds

B. argue for possibility of task
    1. by reducing cardinality to finite
    2. by removing compulsion from task
    3. by appealing to mathematical mechanisms

•.

The catagories are not independent.

Examples illustrate the scheme: Thomson's lamp according to McLaughlin (Al); The Dichotomy according to Alper & Bridger (BI & B2); The Dichotomy according to Aristotle (B2); The Dichotomy according to Sorabji (B2 & B3).

Focusing now on 1ST, a natural classification can be discerned; the two modes of application of 1ST are the "critical" and the "theoretical". In the present work, the critical mode has been employed exclusively, except for reference to the Chryssipus-Benedetti-White resolution of The Arrow, and it is the dominant mode of McLaughlin & Miller.

The critical mode of use of 1ST, as we have seen, is initiated by mapping phenomena representing an infinite task into the system of real numbers, the existence of the map being based upon the mensuration thesis. If one does not judge the task to be compulsory, i.e., if there are no associated phenomena, then the critical mode of IST is not applicable, e.g., Aristotle's and Sorabji's views of The Dichotomy. But, if one does assign phenomena to the task (and accepts the mensuration thesis), the rest of the analysis is mathematical. Here, 1ST is not used to model the world; it is used as a critical tool for demonstrating that certain real numbers are not available for use as measurement labels. Then, by the mensuration thesis, the structures in the world supposedly corresponding to these labels do not exist (as observable). Belief in the soundness of deductions within 1ST is founded on the theorem that 1ST is consistent if ZFC is consistent.

The relationships between the critical mode of 1ST and the taxonomy above are straightforward: Al may be inferred; A2 is not relevant here; B1 may be inferred; B2 cannot be inferred (it is in the province of philosophical judgment); B3 is in the province of the theoretical, not the critical, mode.

The theoretical mode is in the tradition of Chryssipus-Benedetti-White. It is, in a sense, complementary to the critical mode because it uses, as elements of explanation, those mathematical objects which have been rejected as being not observable. Should one feel uncomfortable with staccato motion -- discontinuous leaps from one observable point to the next -- then intermediate, unobservable infinitesimal moves can be postulated in order to restore a sort of continuity to motion. One might, though, prefer a richer philosophical theory than that provided by infinitesimals. For example, Alfred North Whitehead's (1 978) "actual entities" are "drops of experience" which provide a more complex developmental sequence than advance by infinitesimal increments (although the actual entities could be hosted within infinitesimal intervals).

A paper by Alper & Bridger (1 997) rejects the use of 1ST for analyzing Zeno's paradoxes on the ground that it is not "intuitive". They base their objections on developments in McLaughlin & Miller (1 992). The issues raised in their paper provide a setting for further discussion of the philosophical status of IST with regard to Eleaticism and larger domains.

The basic points which Alper & Bridger seek to establish seem to be: 1) IST is flawed as a tool for analyzing phenomena, primarily because its use of "finite" is not intuitive or in accord with experience, and 2) a resolution of The Dichotomy is better obtained by denying that it is compulsory and, in any case, quantum-theoretic considerations reduce it to a finite task.

We respond: 1) the mathematical bases of the arguments of Alper & Bridger with regard to finiteness do not support their claims, 2) they have not treated the concept "intuition" carefully enough to be able to distinguish "not intuitive" from "nontraditional", and 3) their suggested resolution of The Dichotomy contains little that is new and advances no predictions about the world. These points will be addressed in turn.

1. Alper & Bridger cite the use of the unlimited-finite set $S = F \cap [0, 1]$ in McLaughlin & Miller as an example of the nonintuitive use of "finite" in 1ST. In support, they claim (p. 153) ". . . there is no particular reason to believe that there is any difference in 'size' between the set [0, 1] and the set S . . ." It is difficult to believe that anyone could endorse this claim after considering the following comparison of four sets.

Set

Property	[0,1]	rationals in [0,1]	members of the Checkpoint sequence	S ⊊ [0,1]
cardinality	uncountably infinite	countably infinite	countably infinite	finite
well ordered	no	no	yes	yes
greatest element	1	1	none	1
limit points in [0,1]*	all	all	1	none
dense in [0,1]*	yes	yes	no	no
Lebesgue measure	1	0	o	0

● with respect to the Euclidean topology relativized to [0, 1]

Moreover, Alper & Bridger express concern over the fact that S, and other unlimited-finite sets, have infinite cardinality when viewed externally (through a model-theoretic interpretation). But this is not the proposed domain of interpretation for 1ST, for analysis of Eleaticism. Georg Cantor's keystone achievement in creating set theory was to show, through his diagonal argument, that there exists more than one "size" (cardinality) of infinite set. In particular, the real numbers are uncountably infinite while the rational numbers are countably infinite. Nonetheless, the Lowenheim-Skolem theorem permits a demonstration, under a suitable model-theoretic interpretation, that the real numbers have a countable model. (In fact, the theorem shows that all countable theories which have models have countable models.) The dependence of cardinality on the theory within which it is defined is not idiosyncratic and confined to 1ST; it is a general fact. We have chosen the syntactic view of 1ST, its natural setting, and are not distracted by possible semantic excursions. Within the domain of exercise, S and [0, 1] bear no resemblance to one another.

2. Alper & Bridger appeal to our experience and intuition in order that we might conclude, with them, that the definition of finiteness within 1ST makes this version of nonstandard analysis unsuited for analysis of Eleaticism. Although one must acknowledge *de gustibus non est disputandum* with respect to a personal judgment, the concept of intuition of mathematically-defined entities has an extensive history and literature. “Intuition” is not itself an intuitive idea and must be clarified before it can be utilized effectively.

For Aristotle, *nous* is the faculty of intellectual intuition and brings knowledge of *archai* (first principles). It is complemented by *epistemē* (scientific knowledge), which is discursive in nature and is the other procedure for arriving at truth. Nous is not just a technical term within the Aristotelian philosophy but, as Guthrie (1 981) observes, “As far back as Homer it means seeing *and recognizing,* or suddenly grasping, through an act of sensation, the realities of a situation. *Nous* was for Aristotle as for all Greeks the highest of our faculties . . . that on which depends our knowledge of the basic principles or *archai* of deductive science."

The Greek world may have been comfortable with the concept of *nous*, but our acquaintance with axiomatic systems goes well beyond the first principles of (Euclidean) geometry and Aristotelian logic. The multiplicity of available axiom sets and deductive systems has not been narrowed through intellectual perception to a "natural" choice. Kurt Gödel (1 983) represents an optimistic school. "But, despite their remoteness from sense-experience, we do have something like a perception of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. 1 don't see any reason why we should have less confidence in this kind of perception, i.e., in mathematical intuition, than in sense-perception". However, Paul Benacerraf (1973), in considering Gödel's faith, is concerned that the analogy with sense perception is flawed because missing is "an account of the link between our cognitive faculties and the objects known".

Intellectual perception with respect to the *archai* in joint domains of physics and mathematics is also an uncertain cognitive activity, as historical evidence shows. The development of non-Euclidean geometries by Johannes Bolyai (1802-60), Nicolai Lobachevsky (1793-

1856), and Bernhard Riemann (1826 -66) wasdisturbing to a tradition which assumed Euclidean geometry in three dimensions provided the correct mathematical description of space. Hermann Lotze (1887), in developing a theory of space, expresses his traditionally-based convictions. "1 cannot believe that any skill in analysis can compensate for this misconception in the ideas; alleged spaces of such structure that in one part of them they would not be able to receive, without stretching or change size, a figure which they could so receive in another, can only be conceived as real shells or walls, endowed with such forces of resistance as to hinder the entrance of an approaching real figure, but inevitably doomed to be shattered by its more violent impact. I trust that on this point philosophy will not allow itself to be imposed upon by mathematics; space of absolutely uniform fabric will always seem to philosophy the one standard by the assumption of which all these other figures become intelligible to it." The slippery nature of judgments based upon intuition could not be better illustrated. Lotze's prerelativistic recourse to intuition for evaluating the merits of competing mathematical models of physical space, that fundamental arena of human experience, would seem to have been as methodologically valid as Alper & Bridger's views on cardinality and experience.

Is the intuition which Alper & Bridger espouse finely enough developed to distinguished between 1ST and other intellectual systems? We cannot know the answer since they have not made explicit the principles which they employ; presumably they are depending upon a common understanding. But, as the literature shows, this consensus does not exist; one is forced to declare the basis of one's scheme of intuition. Nonetheless, from their critique, some of which was reviewed above in 1, one may discover an unintentional characterization of 1ST: "nontraditional". Advances in our understanding of those fundamental objects, the real line and the Euclidean plane, have historically encountered resistance reserved for the (at the time) nontraditional. Consider the string of pejorative adjectives, "negative", "irrational", "imaginary", "transcendental" (Alper & Bridger employ "mystical" at one point), and compare these with "natural" numbers.

In conclusion, we submit that Alper & Bridger have conflated "not intuitive" or "not in accord with experience" with "nontraditional" or "not customary".

The case for 1ST and the critical mensuration thesis rests, as we have said earlier, on comprehensiveness: dysfunctionality of Thomson's lamp; disabling of Zeno's paradoxes of motion; commentaries on Kant's First Antinomy; Black's ball; Malament-Hogarth spacetimes; and extensions beyond Eleaticism to granularity of space & time, staccato motion, and the necessity of some degree of physical law. However, if we were to attempt a justification for the use of 1ST based on intuition or *now,* a prolegomenon for clarifying the nature of this faculty would rest upon the one principle which clearly emerges from centuries of Eleaticism: it is difficult to apprehend an infinite set. Then, justification of the IST-based approach would consist of calling attention to two items: 1) every infinite set contains a nonstandard object, and 2) nonstandard objects are not accessible.

3. The resolution of the paradoxes of motion suggested by Alper & Bridger blends a noncompulsory view ("B2" in the taxonomy) with reduction to finite cardinality ('(Bl "). The latter strategy is accomplished through acknowledgment of the role of quantum mechanics within Eleaticism. The approach is soundly implemented, and the authors demonstrate good knowledge of the literature.

Their resolution, when considered against the background of Eleaticism, passes the test of comprehensiveness. The quantum theory alone, at least at scales greater than the Planck length, places severe constraints on infinite tasks. (Below this scale, approximately 10^{-33} cm, there is no particular reason to expect the quantum theory, as presently formulated, to describe the physical world,) However, there is little that is new in the resolution of Alper & Bridger, and it is not comprehensive in the sense of

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extending beyond Eleaticism and producing insights related to foundational physics or the philosophy of science.

This concludes our discussion of the work by Alper & Bridger, and we summarize the prospects for Eleaticism and the role of the present approach within that area of research

From another dichotomy, either it is possible to probe at scales below the Planck length (to observe pulses from a limited-finite Thomson's lamp or early progress in The Dichotomy) or it is not; the disposition of the disjunction is a scientific question. If the uncertainty principle does represent an impenetrable epistemological barrier, then it is, among other things, the instantiation of the "granularity" shown by critical use of 1ST. (Recall that the scale of this granularity is measured by a standard real number greater than zero and not by an infinitesimal quantity.) In either case, Eleaticism remains relevant for investigating the structure of a portion of the world.

The mensuration thesis is the connection between the domain of phenomena and mathematics: no perception is beyond the possibility of description, and each description can be rendered in terms of (standard) real numbers. One might question the validity of the principle when mental phenomena are "perceived" through introspection. An attempt could be made to save the mensuration thesis for this interior domain by insisting that mental states are always representable through the medium of language, and, then, we are essentially done. Alternatively, it is feasible to postulate that a human organism can be represented as a scientific object and, ultimately, asuperposition ofquantum-mechanical states. In this case, to extend the mensuration thesis, we would come down on the mind-as-epiphenomenon side of the mind-body problem.

The nature of language and the nature of mind each lie well beyond the scope of these notes, and we rest content with the knowledge that reasonable options exist for the plausibility of the mensuration thesis over a wide range of experience.

In summary, a list of far-reaching effects, beginning with the dysfunctionality of Thomson’s lamp, can be compiled because of the interplay between two factors.

1. The real numbers are a comprehensive descriptive device, through measurement, for characterizing phenomena (mensuration thesis).

2. The real numbers can be examined for their epistemological potential through the critical use of internal set theory.

For approaches to Eleaticism, the relevance of the first factor, to particular problems, is determined through philosophical judgment, The second factor is implemented largely through mathematical technique, but, when the theoretical mode is used instead, philosophical judgment resumes importance.

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