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HALO ORBIT GENERATION USING THE CENTER MANIFOLD

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In recent years, halo orbits around the Sun-Earth Lagrange points, L_1 and L_2, have become extremely popular for NASA missions due to the ideal observation environment and iow launch energy (C_3=-0.6). Traditional methods for the generation of halo orbits require an initial guess provided by some high order analytic expansion. A separate expansion may be required for different systems due to differences in perturbations and small parameter assumptions. Since periodic orbits and quasiperiodic orbits live on the center manifold of the Lagrange point, approximations of the center manifold provide good initial conditions for halo orbit generation. Such a method relies solely on the dynamics of the three body problem and is uniformly applicable for all Sun-Planet and Planet-Moon systems. The approximation of the center manifold is achieved by linearizing the equations of motion of the circular restricted three body problem at the collinear Lagrange points. Using differential correction, a halo orbit is quickly found.

INTRODUCTION

The numerical generation of halo orbits about the collinear libration points require an initial first guess. Typically, this is provided by a high order expansion using the Linstedt-Poincare method (Ref. 1). For the generation of the stable and unstable manifolds of a fixed point, the eigenvectors of the linearized equation at the equilibrium point are used as a first guess (Ref. 2). In this paper, we explore the use of the eigenvectors of the imaginary eigenvalues of the linearized equation at the libration points to generate periodic orbits.

APPROXIMATION OF THE CENTER MANiFOLD

The equations of motion for the Circular Restricted Three Body Problem in normalized rotating coordinates are:

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$$x'' - 2y' = \partial U/\partial x \tag{1}$$
$$y'' + 2x' = \partial U/\partial y$$
$$z'' = \partial U/\partial y$$

where

$$U = 1/2(x^2 + y^2) + (1-\mu)/r + \mu/R$$
$$r^2 = (x + \mu)^2 + y^2 + z^2$$
$$R^2 = (x - 1 + \mu)^2 + y^2 + z^2$$
$$\mu = \text{normalized mass of earth.}$$

This system has 5 equilibrium points, labeled L_1 through L_5 by convention. We are interested in the orbits around the unstable collinear points, L_1, L_2, and L_3. The linearized equations for this system are:

$$q' = M q \tag{2}$$

where

$$q = (x, y, z, x', y', z')^T$$

$$M = \begin{pmatrix} 0 & I \\ U_{xx} & 2\Omega \end{pmatrix}$$

M is a 6x6 matrix with four 3x3 submatrices where

$$0 = \text{zero matrix}$$
$$I = \text{identity matrix}$$
$$\Omega = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and U_{xx} is the matrix of second partials of U. See Szebehely, Ref. 3 For derivation and analysis.

To facilitate the discussion, we define a few concepts from invariant manifold theory. See Wiggins, Ref. 4 for more details. An invariant manifold, S, is a high-dimensional surface composed of solutions of eq. (1). Given any point, q, on S, the solution through q is contained in S. Hence, S is "invariant" under the flow of the solutions. The most familiar examples are fixed points and periodic orbits. Associated with fixed points, are other types of invariant manifolds: the stable manifold, the unstable manifold, and the center manifold of the fixed point. These manifolds partition the phase space into regions with particular dynamics which can be exploited for mission design purposes. For example, the stable manifold has been used to generate launch trajectories from earth to L_1, and L_2; the unstable manifold has been used to generate transfer trajectories between L_1 and L_2, and return trajectories between L_1, L_2 and the earth (see Barden, Howell, & Lo, Ref. 5). For this discussion we focus on the center manifold.

The center manifold of L_1 is a four dimensional manifold in R^6 where the periodic and quasiperiodic orbits around L_1 live in the phase space. It can be approximated near L_1 by the eigenvectors of the four imaginary eigenvalues of the matrix M in eq. (2). For the Sun-Earth L_1 (mass parameter μ = 3.003480924985e-6, the eigenvalues and their corresponding eigenvectors are given in Table 1:

Table 1
Imaginary Eigenvalues and their Eigenvectors at L_1

$\lambda 1$ 2.0864i
e1 0.1279, 0.4129i, 0.0000, 0.2668i, -0,8614, 0.0000

$\lambda 2$ -2.0863 i
e2 0.1279, -0.4129i, 0.0000, -0.2668i, -0.8614, 0.0000

$\lambda 3$ 1.7495206 i
e3 0.0000, 0.0000, 0,4962, 0.0000, 0.0000, 0.8682i

$\lambda 4$ -1.7495206 i
e4 0.0000, 0.0000, 0.4962, 0,0000, 0.0000, -0.8682i

We note {el, e2}, {e3, e4} are conjugate pairs. Thus by multiplying the pairs with complex conjugate coefficients, we may obtain real vectors. Let EC = Span{ e1, e2, e3, e4 } be the vector subspace spanned by these vectors at L_1. Clearly, e1 and e2 correspond to planar solutions; e3 and e4 correspond to solutions in the z-direction. These are the two modes of solutions.

NUMERICAL RESULTS

Let P denote the fixed point L_1 in R^6 coordinates. Then an initial guess for a periodic orbit around L_1 is provided by:

$$X_0 = P + w1*e1 + w2*e2 + w3*e3 + w4*e4$$

which we write as

$$X_0 = P + w^*e \tag{4}$$

where w = {wI, w2, w3, w4} are coefficients such that {w1, w2} and {w3, w4} are conjugate pairs. Using standard differential correction methods (see Howell, Ref. 6), periodic orbits are easily constructed for w with small norms. We recall this algorithm exploits the fact that the halo-family of periodic orbits about L_1 are symmetric about the xz plane. Hence at the xz-crossing, the initial state must have the following form:

$$X_0 = (x_0, 0, z^*, 0, ye', 0). \tag{5}$$

At the next intersection of the xz plane, X_1 will have the form:

$$X_I = (x_1, 0, z_1, x,', y,', Z,'). \tag{6}$$

In order for the orbit to be symmetric about the xz plane, the x and z velocities of X_1 must be O. Two differential correction schemes are given in Ref. 6: method 1 fixes X_o, method 2 fixes Z_o. With this algorithm, only half an orbit need be integrated to produce the desired periodic orbit.

For planar orbits, Howell's algorithm is easily adapted. Eqs. (5) and (6) become:

$$\begin{aligned} X_0 &= (x_0, 0, 0, ye') \\ X_1 &= (x_1, 0, x_1', y,'). \end{aligned} \tag{7}$$

To generate a periodic orbit symmetric about the xz-plane, y_I' must be eliminated by differential correction, Figure 1 presents the class of planar orbits for w = p (1, 1,0, O), for p = 1 e-6, 1 e-5, 1 e-4, 1 e-3 generated by fixing z_0. Note, the orbit for the p = 1 e-6 case is so small it does not appear on the plot.

Figure 2 presents the halo orbit and its intermediate orbits from the differential correction fixing Z. for w = p (1, 1, 1, 1), where p = 1 e-6. The final orbit has y, z amplitudes of {4e-3, 1 e-6} as compared with p. While the z-amplitude is small, it is non-zero, indicating it as one of the first halo orbits. When we perform the differential correction for the same w above, freeing Z. but fixing X., the final orbit has y, z amplitudes of{8e-7,2e-16}. Now Z. = -1e-6 for this w. Thus this correction process is pushing the orbit back into the xy-plane.

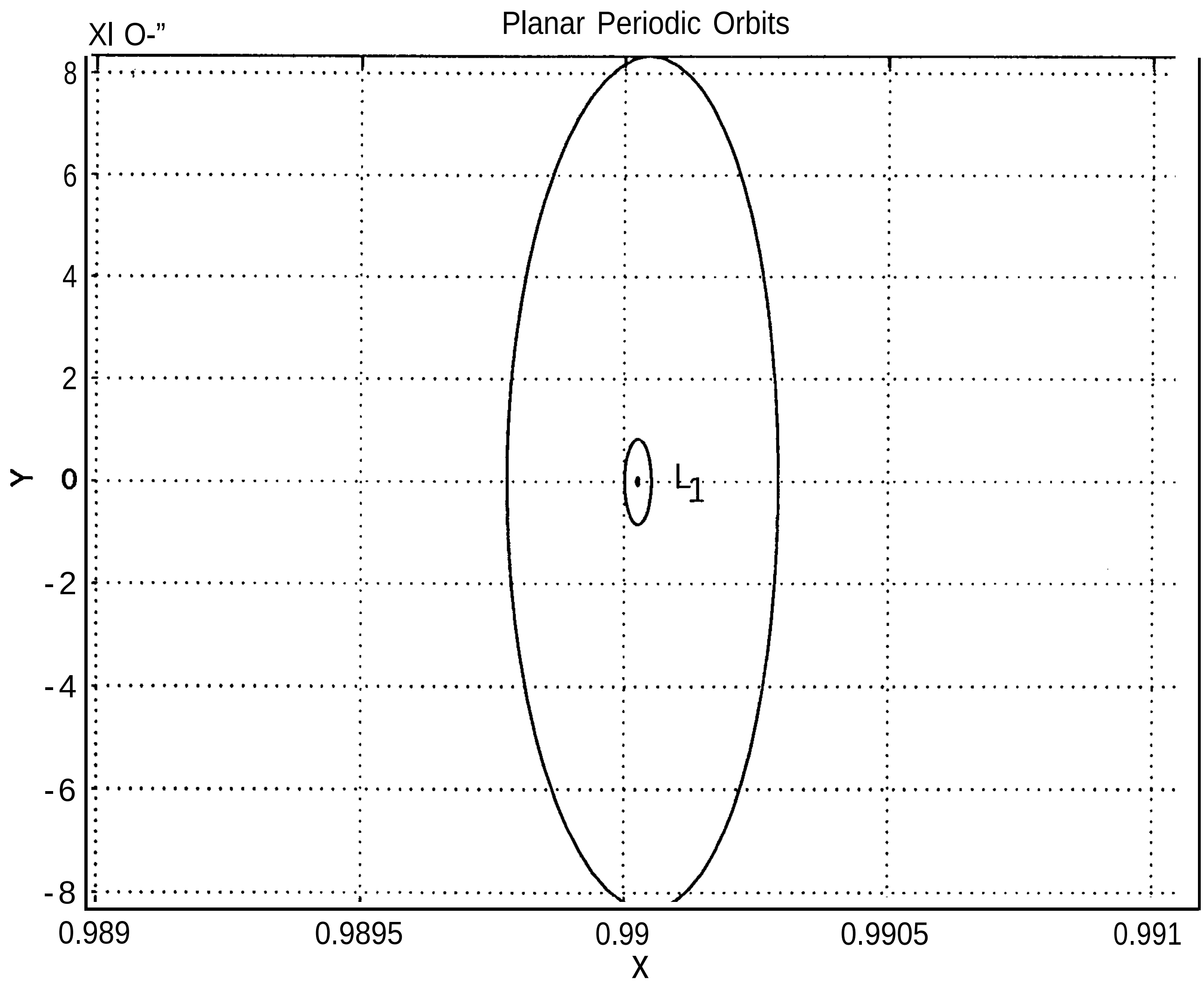

Figure 1. Planar Periodic Orbits Generated with Center Manifold Approximations

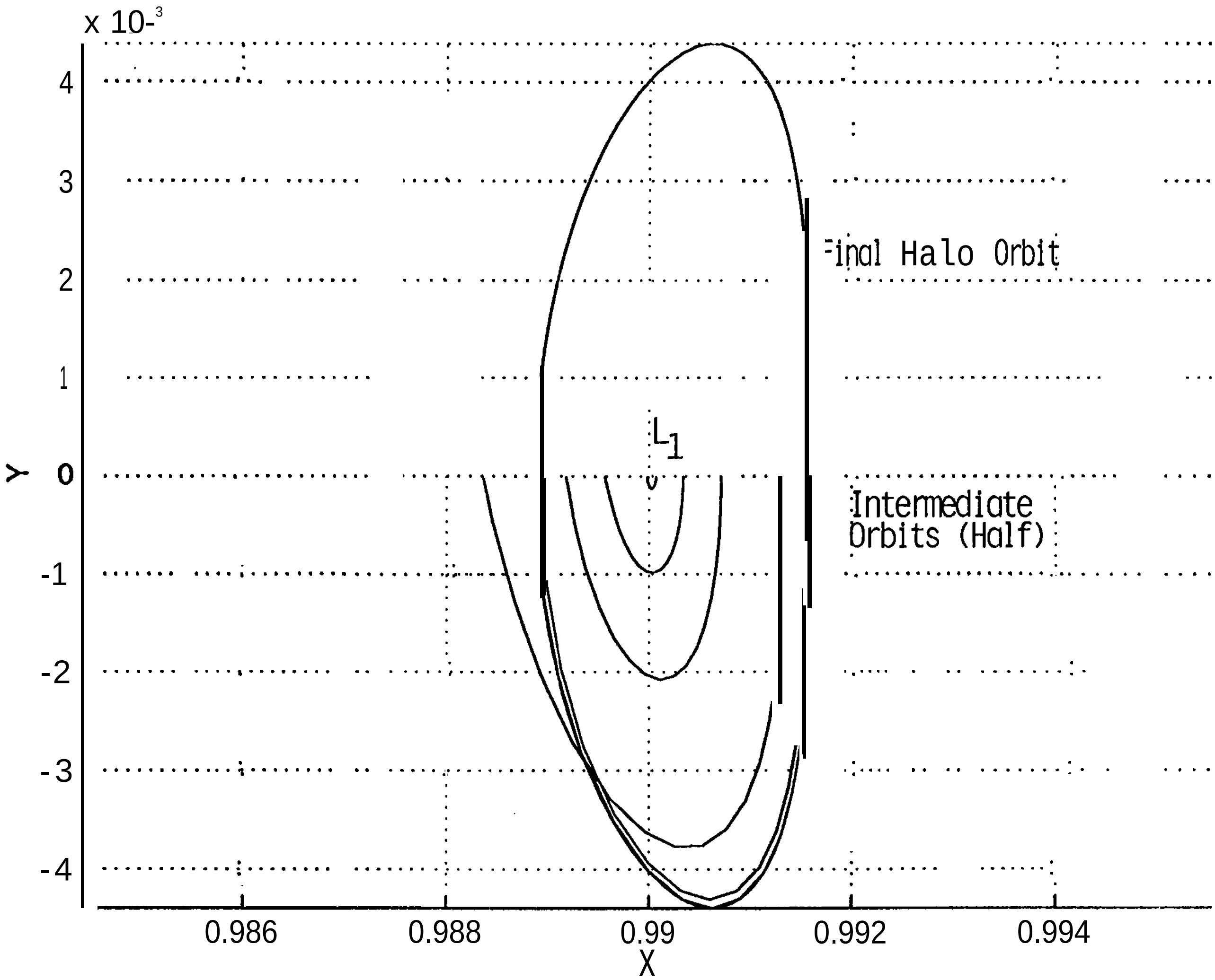

Figure 2. Halo Orbit and Intermediate Orbits from Differential Corrections

It is well known that halo orbits are a non-linear phenomena and do not occur for small y-amplitudes. What is surprising is that this small magnitude linear approximation is able to generate a halo orbit. We need to understand what exactly is going on with this leap-frogging increase in the y-amplitude from 1 e-6 to 4e-3.

For w = p(O, O, 1, 1), an almost straight-line up-and-down motion in the z-direction results but is extremely unstable numerically.

CONCLUSIONS

The linear approximation of the center manifold at L_1 provides a very simple method for generating initial guesses for periodic orbits about L_1. Surprisingly, it is able to produce halo orbits with small z-amplitudes. This is a reminder that despite the large distance the halo orbits are away from L,, they really live on the center manifold of L_1.

Using the x-axis control developed by Dunham, Ref. 7, the linear approximation in eq. (4) should be able to produce small amplitude lissajous orbits with prescribed amplitudes using differential correction. In this approach, one controls the x-coordinate of the orbit as it crosses the xz-plane.

In order to obtain halo orbits with large z-amplitudes, higher order approximations of the center manifold are required. Instead of the Linstedt-Poincare expansion, perhaps the partial differential equation formulation of the center manifold can provide the required globalization of the manifold. This, of course, is a much more difficult problem than the series expansion. But what is gained is a global solution space whose geometry is only hinted at by the expansions. In addition to the visualization of the manifold, it provides quantitative estimates of the periodic and quasiperiodic orbits. This knowledge would allow mission analysts to know where these orbits may be found in the phase space.

Invariant manifolds provide the only real global handle mission analysts have in this extremely complicated phase space, They provide geometric insight as well as quantitative information about the orbits. At the 'QUO VADIS' Dynamical Systems Conference held at UCLA in 1996, Moser (Ref. 8) stated that the numerical computation of invariant manifolds for high dimensional systems is one of the key challenges in dynamical systems today. We believe the numerical calculation of these objects is the foundation for the automatic generation of trajectories in this regime of the three body problem. The applications to mission analysis are numerous: from the launch to halo orbit insertion problem, to station keeping, formation flying around L_1 and L_2, and low-thrust control. Each of these problems would greatly benefit from a deeper understanding of the invariant manifolds around the Lagrange points.

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