Capacity of PPM on Gaussian and Webb Channels[1]

S. Dolinar, D. Divsalar, J. Hamkins, F. Pollara
Jet Propulsion laboratory, California Institute of Technology, Pasadena, California

Abstract

This paper computes and compares the capacities of M-ary PPM on various idealized channels that approximate the optical communication channel: (1) the standard additive white Gaussian noise (AWGN) channel; (2) a more general AWGN channel (AWGN2) allowing different variances in signal and noise slots; (3) a Webb-distributed channel (Webb2); (4) a Webb+Gaussian channel, modeling Gaussian thermal noise added to Webb-distributed channel outputs. We were able to define a suitable bit-normalized SNR parameter ρ_b such that all of these channels with soft outputs yield brick-wall thresholds on the minimum acceptable value of ρ_b above which reliable communication is theoretically possible and below which it is not possible. Furthermore, under all of these models with soft channel outputs, the bit-SNR thresholds for different values of M differ from each other by the "simplex-to-orthogonal penalty" $\frac{M-1}{M}$. Under both the AWGN2 and Webb2 models, the gap between the capacities of hard- and soft-output channels is about 3 dB at the (nonzero) code rate giving the optimum hard-output bit-SNR.

1 Introduction

In an optical communication system using M-ary pulse position modulation (PPM) and an avalanche photodiode (APD) detector, the number n of photons absorbed is Poisson distributed with mean $\bar{n}$ related to the total optical power $P(t)$ and to the PPM slot time T_s by $\bar{n} = \frac{\eta}{h\nu}\int_0^{T_s} P(t)dt$, where η is the detector's quantum efficiency and $h\nu$ is the photon energy. For PPM signaling, the mean number of photons $\bar{n}$ absorbed in a given PPM slot depends on whether the signal is present or absent in that slot. In response to n absorbed photons, the APD generates an "avalanche" of q electrons with a complicated conditional probability distribution derived by McIntyre (see reference in [1]), and the probability mass function $p(q)$ is obtained by averaging this conditional probability over the Poisson-distributed n. Alternatively, $p(q)$ can be approximated by a simpler continuous probability density derived by Webb (see reference in [1]):

$$p(q) = \frac{1}{\sqrt{2\pi\bar{n}G^2F}}\left(1+\frac{(q-G\bar{n})(F-1)}{GF\bar{n}}\right)^{-3/2}\exp\left(-\frac{(q-G\bar{n})^2}{2\bar{n}G^2F\left(1+\frac{(q-G\bar{n})(F-1)}{GF\bar{n}}\right)}\right),\quad q > \frac{-G\bar{n}}{F-1} \tag{1}$$

where G is the APD gain, and $F = k_{eff}G + (2-1/G)(1-k_{eff})$ is an excess noise factor, given in terms of the gain and the ionization ratio k_{eff}. The Webb model for PPM signaling (here called Webb2) uses the density in eq. (1) twice: once using the average number $\bar{n}_1$ of photons in the signal slot, and a second time using the average number $\bar{n}_0$ of photons in the $M-1$ non-signal slots. The final output of the APD is then modeled as a sum of an electrical current due to the Webb-distributed electrons, along with Gaussian thermal and surface leakage currents (here called the Webb+Gaussian channel).

It is known [1] that the Webb density is well approximated by a Gaussian away from its tails, and that the approximation accuracy improves as $\bar{n}_1$ and $\bar{n}_0$ get large. Our objective in this paper is to develop an understanding of the role of various optical parameters on the capacity of an optical communication system, and to this end we compute and compare the capacities of various idealized channels which might be used to approximate the optical communication channel: (1) the standard additive white Gaussian noise (AWGN) channel; (2) a more general AWGN channel (AWGN2) allowing different variances in signal

[1]This research was carried out at the Jet Propulsion Laboratory, California Institute of Technology, under a contract with NASA.

and noise slots; (3) a Webb-distributed channel (Webb2) modeling the output of an APD in the absence of thermal noise and surface leakage current; and (4) a Webb+Gaussian channel, modeling Gaussian thermal noise and surface leakage current added to Webb2 channel outputs. We also compare the capacities achievable with soft- and hard-decision channel outputs. The appeal of using soft decisions lies in the ability to take advantage of better performing codes (e.g., turbo codes), which admit soft decoding algorithms.

2 Capacity of M-ary PPM on the AWGN Channel

First we consider the case where the signal $\mathbf{x}$ is transmitted on a standard AWGN channel with noise spectral density $N_0/2$. Because of the symmetry of orthogonal signals and of the AWGN channel, capacity is achieved with an equiprobable M-ary source distribution $p(\mathbf{x} = \mathbf{x}_j) = 1/M$, $\forall \mathbf{x}_j \in \mathcal{S}$, where $\mathcal{S} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_M\}$, and the capacity reduces to

$$C = \int_{\mathbf{y}} p(\mathbf{y}|\mathbf{x}_1) \log_2 \left(\frac{p(\mathbf{y}|\mathbf{x}_1)}{\frac{1}{M} \sum_{\mathbf{x}_j \in \mathcal{S}} p(\mathbf{y}|\mathbf{x}_j)} \right) d\mathbf{y} \tag{2}$$

where $\mathbf{y} = (y_1, \ldots, y_M)$ is the received vector. This capacity expression assumes no restriction on the channel output. For orthogonal codes on the AWGN channel, $\mathbf{x}_j = (x_{j1}, \ldots, x_{jM}) = (0, 0, 0, \ldots, \sqrt{\rho}, 0, \ldots, 0)$, where $\sqrt{\rho} = \sqrt{2E_s/N_0}$ is in position j, and E_s refers to the energy per M-dimensional symbol. We have [2]

$$p(\mathbf{y}|\mathbf{x}_j) = \prod_{i=1}^{M} \frac{e^{-(y_i - x_{ji})^2/2}}{\sqrt{2\pi}} \tag{3}$$

and the capacity of orthogonal signaling follows from (2) as

$$\boxed{C = \log_2 M - E_{\mathbf{y}|\mathbf{x}_1} \log_2 \left[\sum_{i=1}^{M} e^{\sqrt{\rho}(y_i - y_1)} \right]} \tag{4}$$

Figure 1 shows this capacity for various PPM dimensions M as a function of the minimum required *bit-energy-to-noise ratio*, $\rho_b = E_b/N_0 = \frac{E_s/N_0}{C} = \frac{\rho/2}{C}$. To compute C for large dimensions M, it is necessary to resort to Monte Carlo methods. (Similar methods were used for the computations in [3, Appendix I].) Note that all of the curves approach "brick-wall" thresholds on E_b/N_0, below which the capacity falls to zero.

3 Capacity of M-ary PPM on a More General Gaussian Channel

Now we extend the analysis to cover a "double Gaussian" problem (here called AWGN2), related more directly to the PPM optical model and characterized by different means and variances depending on whether the signal is present or absent. Given a transmitted signal $\mathbf{x} = \mathbf{x}_j$, the components of the received vector $\mathbf{y}$ are taken to be conditionally independent Gaussian random variables, identically distributed except for y_j:

$$\begin{cases} y_i & \text{is } N(m_0, \sigma_0^2), \quad i \neq j \quad \text{(signal absent)} \\ y_j & \text{is } N(m_1, \sigma_1^2) \qquad\qquad \text{(signal present)} \end{cases} \tag{5}$$

By symmetry of the signal constellation $\mathcal{S}$, capacity can again be evaluated by eq. (2), and we have

$$p(\mathbf{y}|\mathbf{x}_j) = \left(\frac{1}{2\pi\sigma_0}\right)^{\frac{M-1}{2}} \left(\frac{1}{2\pi\sigma_1}\right)^{\frac{1}{2}} e^{-\frac{1}{2}(y_j - m_1)^2/\sigma_1^2} \prod_{\substack{i=1 \\ i \neq j}}^{M} e^{-\frac{1}{2}(y_i - m_0)^2/\sigma_0^2} \tag{6}$$

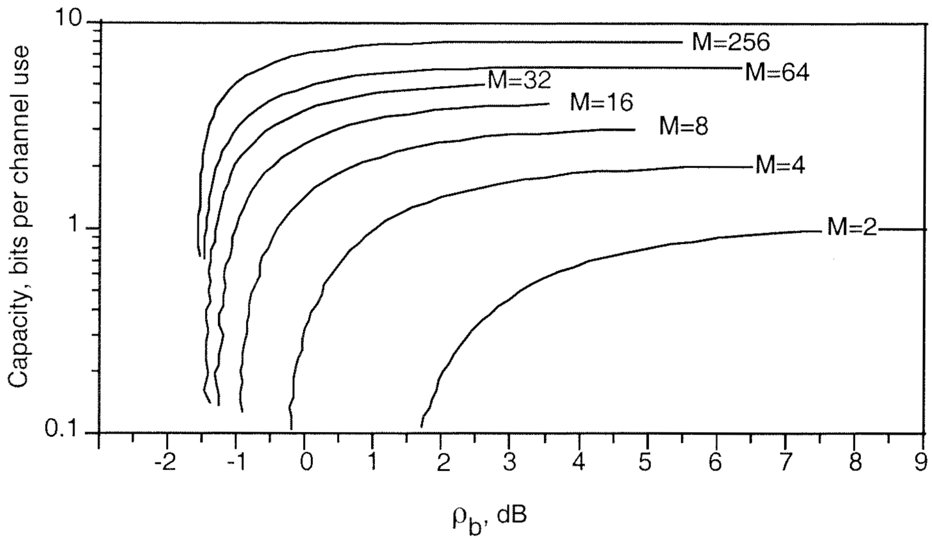

Figure 1: Capacity of M-PPM (or any other M-ary orthogonal signaling) on the AWGN channel.

Even though the channel model uses four parameters, the capacity is determined by just two parameters, and can be written as

$$\boxed{C = \log_2 M - E_{\mathbf{v}} \log_2 \left[\sum_{i=1}^{M} e^{\sqrt{\rho_0}(v_i - v_1)/(1+\alpha\rho_0)} \right]} \tag{7}$$

where $\rho_0 = (m_1 - m_0)^2/\sigma_0^2$, $\alpha = (\sigma_1^2 - \sigma_0^2)/(m_1 - m_0)^2$, $\mathbf{v} = (v_1, \ldots, v_M)$, $v_i = u_i + \frac{1}{2}u_i^2\alpha\sqrt{\rho_0}$, and $\{u_1, \ldots, u_M\}$ are independent Gaussian random variables: u_1 is $N(\sqrt{\rho_0}, 1 + \alpha\rho_0)$ and u_i is $N(0, 1)$ for $i > 1$. Note that this equation reduces to the standard AWGN capacity for orthogonal signals (eq. 4), when $\alpha \to 0$, i.e., when the variances in the signal and non-signal slots become equal. Figure 2 shows the capacity obtained by Monte Carlo integration for the AWGN2 model, plotted for different values of α versus a bit-normalized SNR parameter defined by $\rho_b = \frac{\rho_0/2}{C}$.

4 M-ary PPM Capacity on the Webb Channel

The capacity of M-PPM on a Webb channel is conveniently computed by substituting a standardized (scaled-and-translated) Webb random variable w for the Webb-distributed electron count q in (1). Defining $q = G\bar{n} + w\sqrt{\bar{n}G^2F}$, the probability density for the standardized Webb random variable w simplifies to

$$p(w) = \frac{1}{\sqrt{2\pi}}(1 + \beta w)^{-3/2} e^{-w^2/2(1+\beta w)}, \quad 1 + \beta w > 0 \tag{8}$$

where $\beta = (F^{1/2} - F^{-1/2})/\bar{n}^{1/2}$. Note that this standardized Webb probability reduces exactly to a standardized Gaussian when the parameter $\beta \to 0$.

The optical PPM problem is modeled as a "double Webb" channel (Webb2), for which the random number of electrons produced by the APD is given by $q_1 = G\bar{n}_1 + w_j\sqrt{\bar{n}_1 G^2 F}$ in the signal slot j, and $q_0 = G\bar{n}_0 + w_i\sqrt{\bar{n}_0 G^2 F}$ in the non-signal slots $i \neq j$, where $\{w_1, \ldots, w_n\}$ are independent standardized Webb random variables with probability density given by (8).

The capacity of the Webb2 channel is obtained by Monte Carlo integration of (2) using the appropriate Webb densities for the channel transition probabilities $p(\mathbf{y}|\mathbf{x})$. Figure 3 plots the capacity of 256-PPM on a Webb2 channel for various values of $\bar{n}_s = \bar{n}_1 - \bar{n}_0$ versus a bit-normalized SNR parameter defined

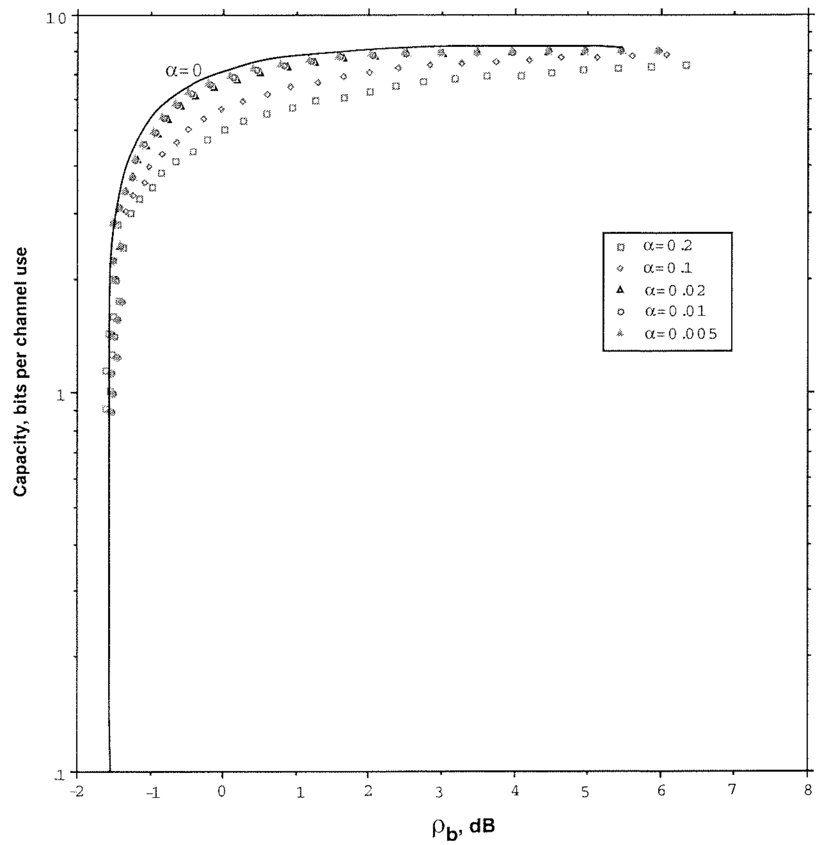

Figure 2: Capacity of the AWGN2 channel for PPM dimension $M = 256$. The case $\alpha = 0$ is equivalent to the standard AWGN channel.

by $\rho_b = \frac{\rho_0/2}{C}$, where $\rho_0 = (\mathrm{E}\{q_1 - q_0\})^2/\mathrm{Var}\{q_0\}$. Note that this definition of ρ_b for the Webb2 channel is exactly analogous to the one given earlier for the AWGN2 channel, and the series of Webb2 capacity curves in Figure 3 approach the capacity curve of the AWGN channel (shown for reference) as $\bar{n}_s$ gets large.

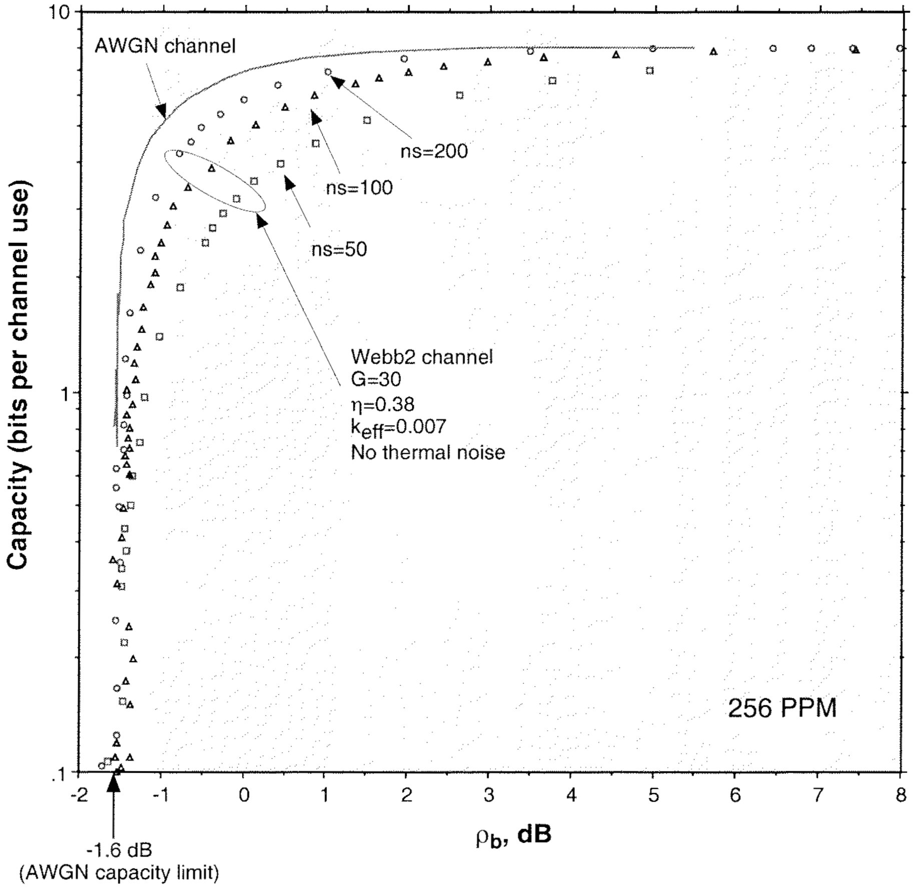

Figure 3: Capacity of 256-PPM for AWGN and Webb2 channels.

Figure 4 plots the Webb2 channel capacities for $\bar{n}_s = 100$ and different values of M. This figure also shows the corresponding M-PPM capacities of the AWGN channel for comparison. Note that the Webb2 capacities, plotted versus ρ_b, all approach the same brick-wall thresholds (within the statistical uncertainties of the Monte Carlo integration) as those characterizing the AWGN channel. Furthermore, for both the AWGN and Webb2 channels, these brick-wall thresholds, for different values of M, are separated by the (logarithm of the) factor $(M-1)/M$, which represents the loss for using orthogonal signals instead of simplex signals. This penalty is as high as 3 dB for using 2-PPM instead of BPSK, but diminishes rapidly for larger M. In the case of the (coherent) AWGN channel, this "simplex-to-orthogonal" penalty can be avoided, and all of the brick-wall thresholds moved to the ultimate capacity limit of -1.59 dB, by substituting simplex signals for the orthogonal PPM signal set. In the case of the (noncoherent) optical channel, such a substitution is not possible in the physical system, but the comparison still gives a simple numerical yardstick for comparing Webb2 channel capacities for different values of M.

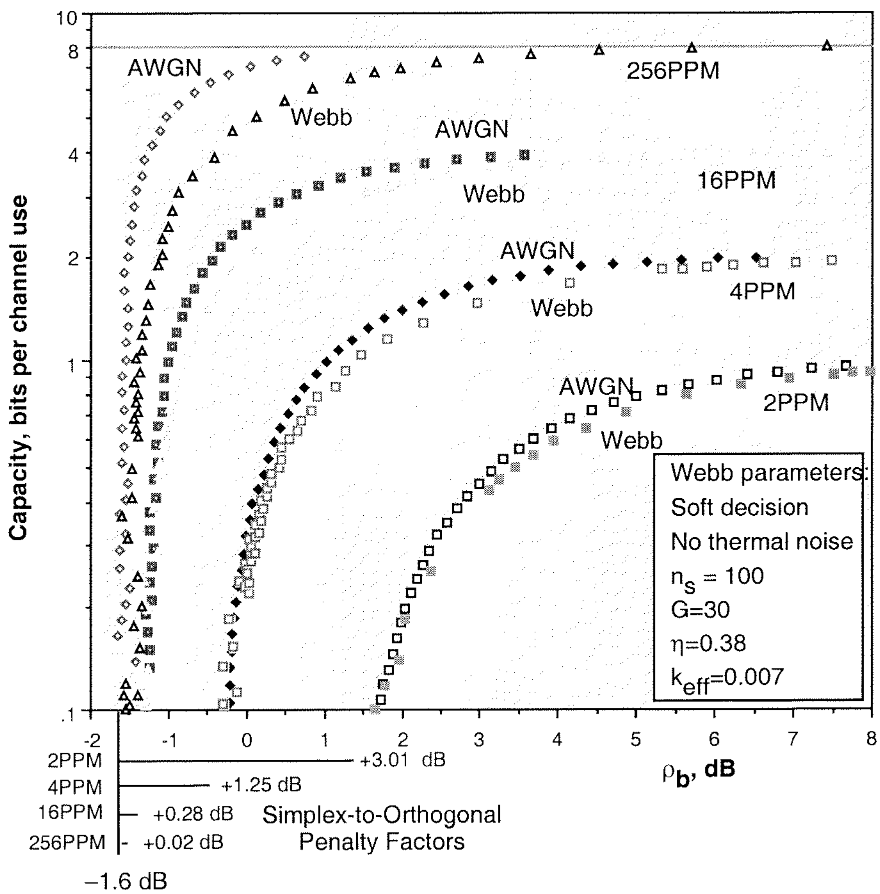

Figure 4: Capacity of AWGN and Webb2 channels for different PPM sizes.

5 Comparison of Hard-Decision and Soft-Decision Capacity for M-ary PPM on the Webb+Gaussian Channel

When a hard-decision detection scheme is used, the decoder operates on PPM symbol decisions from the demodulator, not individual soft counts. The modulator output is the PPM symbol having the maximum slot count. This hard-decision channel is an M-ary input, M-ary output, symmetric channel with capacity given by

$$C = \log_2 M + (1-\epsilon)\log_2(1-\epsilon) + \epsilon \log_2(\frac{\epsilon}{M-1}) \quad \text{bits per channel use,} \tag{9}$$

where ϵ is the probability of incorrect symbol detection:

$$\epsilon = 1 - \int_{-\infty}^{\infty} p_1(x) \left[\int_{-\infty}^{x} p_0(y)dy\right]^{M-1} dx, \tag{10}$$

and $p_1(\cdot)$ and $p_0(\cdot)$ are the channel symbol probability densities for signal and nonsignal slots, respectively.

Figure 5 compares capacities for the hard-output and soft-output AWGN2 channels, for the case of $M = 256$. A similar comparison of capacities is shown in Fig. 6 for the hard-output and soft-output Webb2 channels. The hard-output Webb2 capacity was computed in [4].

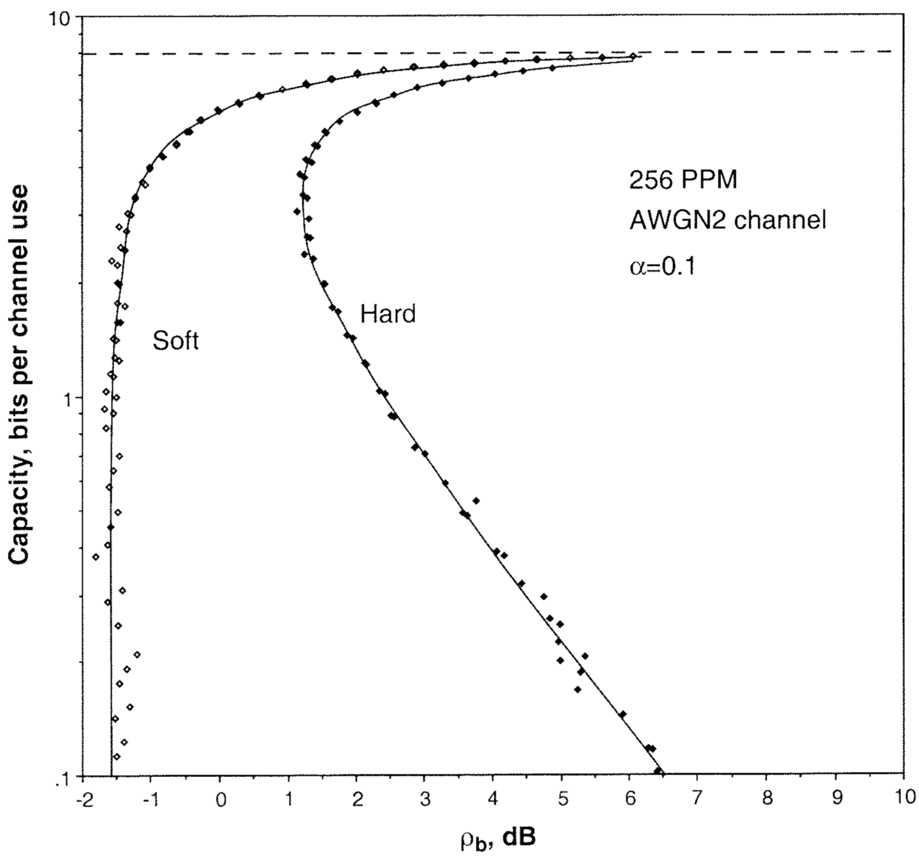

Figure 5: Capacity of 256-PPM on the hard- and soft-output AWGN2 channels.

The capacity curves for both the AWGN2 and the Webb2 channels show that a minimum value of ρ_b is reached at a nonzero code rate. Unlike the soft-output channels, which exhibit monotonically better efficiency in terms of the bit-normalized SNR parameter ρ_b as the code rate (and hence the capacity per channel use) is reduced toward zero, the bit-normalized SNR efficiency of the hard-output channel turns around if the capacity per channel use is lowered below about 4 bits per channel use. This implies that an optimum code rate of about 1/2 will achieve the lowest ρ_b for the hard-output channel, while the soft-output channel achieves lowest ρ_b in the limit as the code rate goes to 0.

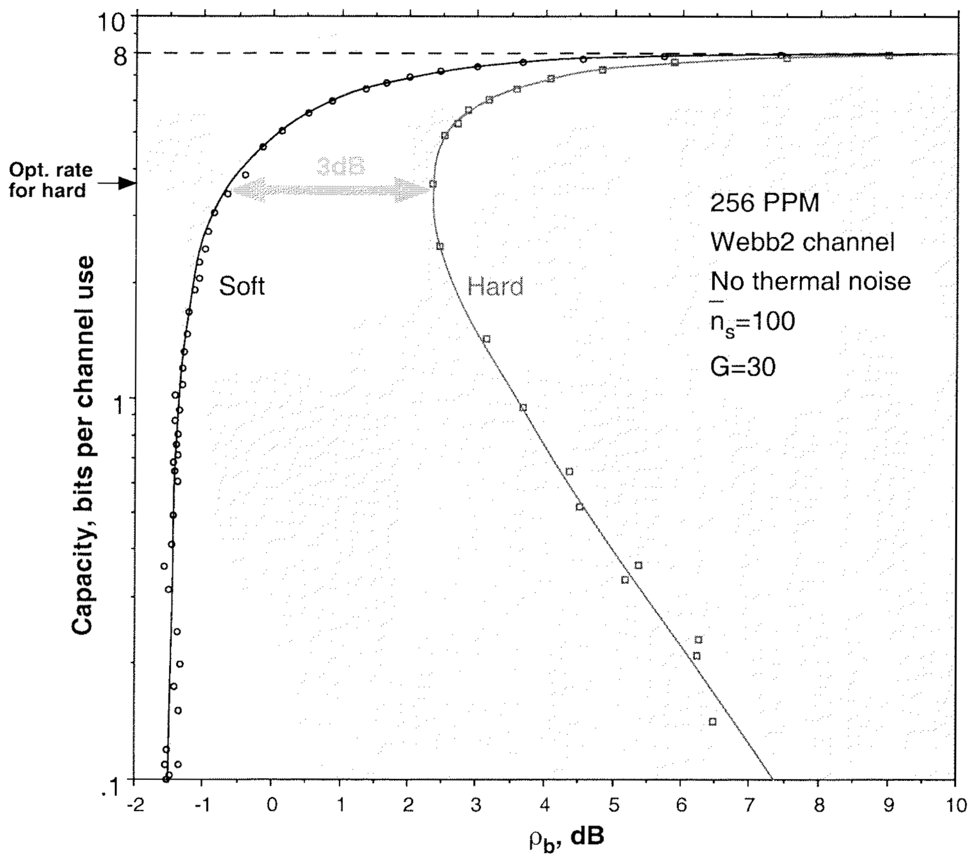

Figure 6: Capacity of 256-PPM on the hard- and soft-output Webb2 channels.

6 Conclusions

This paper has analyzed four idealized channel models that can be used to approximate an APD-detected optical communication channel. We were able to define a suitable bit-normalized SNR parameter ρ_b such that all of these channels with soft outputs yield brick-wall thresholds on the minimum acceptable value of ρ_b above which reliable communication is theoretically possible and below which it is not possible. Furthermore, under all of these models with soft channel outputs, the bit-SNR thresholds for different values of M differ from each other by the "simplex-to-orthogonal penalty" $\frac{M-1}{M}$. Under both the AWGN2 and Webb2 models, the gap between the capacities of hard- and soft-output channels is about 3 dB at the code rate giving the optimum hard-output bit-SNR.

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