A Polynomial-time Algorithm for Solving Certain Classes of Rank Minimization Problem

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Abstract

We present a non-interior point based, polynomial-time, algorithm for solving certain classes of rank minimization problem. Some of the structural properties of the rank minimization problem are also presented.

Keywords: Rank minimization under LMI constraints; Monotone operators; Riccati equation

1 Introduction

The present note grew out of the discussion between the authors on a non-interior point based algorithm for solving a class of rank minimization problem (RMP). The RMP is formulated as follows: given a symmetry preserving linear map M on the space of symmetric matrices, and a particular symmetric matrix Q, solve,

$$\min_X \text{rank } X \tag{1.1}$$

$$-Q + M(X) \geq 0, \tag{1.2}$$

$$X \geq 0, \tag{1.3}$$

where the inequality "$\geq$" is interpreted in the sense of Löwner, i.e., $A \geq B$ signifies that the matrix $A - B$ is positive semi-definite. For a given map M on the space of symmetric matrices and a symmetric matrix Q, the corresponding instance of the rank minimization problem is denoted by RMP(Q, M). In this note we shall always assume that $Q \geq 0$.

Initially, in §2, we restrict our attention to linear maps which have the form,

$$M : X \to X - \sum_i M_i X M_i^T; \tag{1.4}$$

after delineating on some of the structural properties of the corresponding RMP, we propose an algorithm for its solution. In §3 we proceed to make two important generalization of the result presented in §2 which correspond to having a linear map which is induced from an arbitrary monotone linear map, or from the discrete Riccati equation.

The restricted class of RMPs considered in §2 was first studied in [4], and subsequently generalized in [3]. In these works, it was observed that in solving this class of the RMP's, the objective of minimizing the rank can be substituted by minimization of the trace. As a result, the problem could be solved via the recently proposed interior point methods for solving Semi-Definite Programming (SDP) [5].

The purpose of the current note is to show that a simple, non-interior point based solution method for this class of RMPs, as well as for those discussed §3, can be proposed which is more efficient than the interior point based counter-part.

A few words on the notation and some preliminaries. For a matrix A, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ denote its range space and null space, $A^\dagger$ its pseudo-inverse. $\mathbf{SR}^{n\times n}$ denotes the space of real symmetric $n \times n$ matrices; the inner product induced by the trace of the product makes $\mathbf{SR}^{n\times n}$ an inner product space. $\mathbf{SR}_+^{n\times n}$, the space

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of $n \times n$ positive semi-definite matrices, is a pointed convex cone in $\mathbf{SR}^{n\times n}$ which defines an order, making $\mathbf{SR}^{n\times n}$ an ordered vector space. However the resulting ordered vector space is not a vector lattice, since every of its non-empty finite subset does not have a greatest lower bound with respect the positive semi-definite ordering.

Two operations on the cone $\mathbf{SR}_+^{n\times n}$ is defined by rank and trace,

$$\mathbf{rank} \quad : \quad \mathbf{SR}_+^{n\times n} \to \{0,1,\ldots,n\}, \tag{1.5}$$

$$\mathbf{trace} \quad : \quad \mathbf{SR}_+^{n\times n} \to \Re_+. \tag{1.6}$$

We note that trace is convex and rank is not. When a linear matrix inequality is augmented with a rank constraint, the resulting feasibility or optimization problem *usually* becomes NP-hard. For example rank constraints can be used to express integrality of the solution sought in a linear program. Minimizing the trace of a matrix subject to linear matrix inequalities, on the other hand, can in principle, be solved by the recently proposed interior point methods. Although minimizing the trace and the rank subject to LMI constraints are two different classes of problems, we note some obvious relationships between the two; for example, when all eigenvalues of the matrices in the feasible set are either zero or one, then the two optimization problems are equivalent.

2

In this section, we present a sequence of basic lemmas which will lead us to a non-interior point based method for solving (1.1)-(1.3) with M defined as (1.4). Define the feasible set of corresponding RMP as Γ, i.e.,

$$\Gamma := \{X \geq 0 : -Q + X - \sum_i M_i X M_i^T \geq 0\}. \tag{2.7}$$

The solution set of the RMP, consisting of all matrices which have minimum rank in Γ, shall be denoted by Γ^*. Note that

$$\Gamma \neq \emptyset \Rightarrow \Gamma^* \neq \emptyset.$$

Related, but not equivalent to Γ^* is the least element of the set Γ, if one exists.

Definition 2.1 *A subset Λ of $\mathbf{SR}^{n\times n}$ is said to have a least element. if there is a matrix $X \in \Lambda$ such that for every $Y \in \Lambda$, $X \leq Y$. The set of the least elements of Λ shall be denoted by Λ_{inf}.*

Proposition 2.1 *Λ_{inf} is either empty or a singleton.*

Proof: If $\Lambda \neq \emptyset$, suppose that $X, Y \in \Lambda_{\text{inf}}$. Then $X \leq Y$ and $Y \leq X$, thus $X = Y$. ∎

Note that in general, even if $\Gamma \neq \emptyset$, Γ_{inf} can be empty. However for the particular set Γ (2.7), one can show that this can not be the case.

Lemma 2.2 ([4]) *If $\Gamma \neq \emptyset$, then $\Gamma_{\text{inf}} \neq \emptyset$.*

The connection between Γ^* and Γ_{inf} can be established as follows.

Proposition 2.3

$$\Gamma_{\text{inf}} \subseteq \Gamma^*$$

Proof: Recall that if $Z \leq X$, then $\lambda_i(Z) \leq \lambda_i(X)$ $(i = 1,\ldots,n)$, where the eigenvalues are arranged and indexed in a non-increasing order. Suppose that for some matrix $X \in \Gamma$, $\mathbf{rank}\ (X) < \mathbf{rank}\ (Z)$. Then there is a particular index j, such that $\lambda_j(X) = 0$, but $\lambda_j(Z) \neq 0$; however, $\lambda_i(Z) \geq 0$ (for all i), leading to a contradiction. ∎

In the subsequent discussions, we shall assume that $\Gamma \neq \emptyset$. Define the linear map $F : \mathbf{SR}^{n\times n} \to \mathbf{SR}^{n\times n}$ as,

$$F : X \to Q + \sum_i M_i X M_i^T.$$

We now have the following sequence of propositions.

Proposition 2.4 *For all $X \in \Gamma$, $X \geq Q$.*

Proof: If $X \in \Gamma$, then $X \geq Q + \sum_i M_i X M_i^T$ and $\sum_i M_i X M_i \geq 0$. Thus, $X \geq Q$. ∎

Proposition 2.5 *For all $X \in \Gamma$, and for all $n \geq 1$, $F^{(n)}(Q) \leq X$.*

Proof: By induction: for $n = 1$, since $X \geq Q$, $\sum_i M_i X M_i^T \geq \sum_i M_i Q M_i^T$. Thus $Q + \sum_i M_i Q M_i^T \leq Q + \sum_i M_i^T X M_i \leq X$. Therefore $F^{(1)}(Q) \leq X$.

Suppose that the statement of the proposition holds for $n = k$, i.e., $F^{(k)}(Q) \leq X$. Then,

$$\sum_i M_i F^{(k)}(Q) M_i^T \leq \sum_i M_i X M_i^T,$$

and therefore,

$$Q + \sum_i M_i F^{(k)}(Q) M_i^T \leq Q + \sum_i M_i X M_i^T \leq X,$$

i.e., $F^{(k+1)}(Q) \leq X$. ∎

Proposition 2.6

$$F^{(n)}(Q) \leq F^{(n+1)}(Q).$$

Proof: Since $Q \geq 0$, $Q \leq Q + \sum_i M_i Q M_i^T$. Suppose the proportion is true for $n = k$, i.e., $F^{(k)}(Q) \leq F^{(k+1)}(Q)$. Then,

$$\sum_i M_i F^{(k)}(Q) M_i^T \leq M_i F^{(k+1)}(Q) M_i^T$$

$$\Rightarrow Q + \sum_i M_i F^{(k)}(Q) M_i^T \leq Q + M_i F^{(k+1)}(Q) M_i^T,$$

and thus $F^{(k+1)}(Q) \leq F^{(k+2)}(Q)$. ∎

We now recall the following result.

Proposition 2.7 ([G]) *If P_k, Q are symmetric maps such that $P_k \leq Q$ $(k = 1, 2, \ldots)$, and $P_k \leq P_{k+1}$ then the limit*

$$P := \lim_{k \to \infty} P_k$$

exists.

As a direct consequence of this proposition, we obtain the following result.

Proposition 2.8 *The matrix,*

$$X^* := \lim_{k \to \infty} F^{(k)}(Q),$$

exists. Moreover, for all $X \in \Gamma$, $X^* \leq X$.

Lemma 2.9

$$X^* \in \Gamma.$$

Thus $X^* \in \Gamma_{\mathrm{inf}}$.

Proof: Observe that $X^* \geq 0$ by construction. Moreover,

$$Q + \sum_i M_i X^* M_i^T = F(X^*) = X^* \leq X^*.$$

∎

We close this section with the following note. Not only does the above procedure suggests a direct iterative algorithm for determining the least element, and thus the minimal rank matrix of the set Γ, it also suggests the following two questions:

- Can the class of linear maps for which the above procedure is applicable be expanded?
- Can one establish a rate of convergence for the above algorithm? Is the proposed algorithm polynomial-time?

We address both questions in the following section.

3 Two Generalizations

We consider two important generalization of the RMPs considered in §2 for which the above iterative procedure is applicable. We then discuss the rate of convergence and the polynomial-time solvability issues.

Consider the RMP of the form,

$$\min_X \mathbf{rank}\ X \tag{3.8}$$
$$-Q + X - F(X) \geq 0, \tag{3.9}$$
$$X \geq 0, \tag{3.10}$$

where $F : \mathbf{SR}_+^{n \times n} \to \mathbf{SR}_+^{n \times n}$ is either,

1. an arbitrary monotone linear map, i.e., $X \leq Y$ implies $F(X) \leq F(Y)$ or,
2. $F(X) = A^T X (A - B(R + B^T X B)^\dagger B^T) X A$, where $A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$, and $R \geq 0$ (recall the discrete Riccati equation).

Consider the algorithm which was proposed in §2: Let $X(0) = Q$, $X(k+1) = F(X(k)) + Q$, $k \geq 1$.

Lemma 3.1 *Given that the RMP (3.8)-(3.10) is feasible,*

$$X^* := \lim_{k \to \infty} X(k)$$

exists. Moreover, X^* is the solution of the RMP.

Proposition 3.2 *Let $d = \mathbf{rank}\ Q$. Then,*

$$\mathbf{rank}\ \ (X^*) = \mathbf{rank}\ \ (X(n-d))$$

i.e., the proposed iterative algorithm can be terminated after a finite number of steps.

Proposition 3.3 *Given that the RMP (3.8)-(3.10) is feasible, for some $0 < \alpha < 1$ and $b > 0$,*

$$\|X(k) - X^*\| \leq b\, \alpha^k.$$

In fact an iterative procedure, based on the above algorithm, can be constructed which proceeds from $X(2^k)$ to $X(2^{k+1})$ providing a Newton-like convergence.

The proofs of these results are omitted for brevity and the reader is referred to upcoming journal version of this note [2]

On the question of checking the feasibility of the RMP (3.8)-(3.10) (as well as RMPs considered in §2) one can in fact establish the following results.

Let F be as a monotone linear map $\mathbf{SR}_+^{n \times n}$ and $X^* = X(n-d)$. Define,

$$S(X^*) := \{X \in \mathbf{SR}^{n \times n} \mid \mathcal{R}(X) \subseteq \mathcal{R}(X^*)\}.$$

Proposition 3.4 *$S(X^*)$ is F-invariant.*

Let F_S be the restriction of F on $S(X^*)$.

Proposition 3.5 *The RMP (3.8)-(3.10) is feasible if and only if the spectral radius of F_S is strictly less than 1.*

Similar results can be used for case where the linear map F corresponds to the discrete Riccati equation [1].

Let $X^* = X(n-d)$. Write the matrices A and B as,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

where $A_{11}, A_{21} \in \mathcal{N}(X^*)$ and $A_{12}, A_{22} \in \mathcal{R}(X^*)$.

Theorem 3.6 *The RMP (3.8)-(3.10) with F corresponding to the discrete Riccati equation is feasible if and only if the pair (A_{22}, B_2) is stabilizable.*

4 Conclusion

In this note we established some structural and algorithmic results for certain classes of rank minimization problem; as a direct consequence, we were able to propose a simple, polynomial-time algorithm for their solution.

References

[1] L. Gurvits. *Geometric approach to some problems of control, observation, and lumping.* Ph.D. Thesis, Department of Mathematics, Gorky State University, USSR, 1985.

[2] L. Gurvitz and M. Mesbahi Geometric algorithms for solving certain classes of Semi-Definite and Rank Minimization Problems (in preparation).

[3] M. Mesbahi. On the rank minimization problem and its control applications. *Systems and Control Letters*, 33:31–36, 1998.

[4] M. Mesbahi and G. P. Papavassilopoulos. On the rank minimization problem over a positive semidefinite linear matrix inequality. *IEEE Transactions on Automatic Control*, 42(2):239–243, 1997.

[5] Y. Nesterov and A. Nemirovskii. *Interior-Point Polynomial Algorithms in Convex Programming.* SIAM, Philadelphia, 1994.

[6] W. M. Wonham. *Linear Multivariable Control: A Geometric Approach.* Springer-Verlag, New York, 1979.