

# The Gravitational Potential of a Body in Terms of Ellipsoidal Harmonics

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## Abstract

The gravitational potential of a body in terms of spheroidal harmonics is of great practical use for satellite dynamics. Spheroidal harmonic functions can be described as triple products of functions whose arguments are a single spheroidal coordinate; the zonal and tesseral harmonics vanish along lines of constant latitude or longitude, which trace out planes and cones. This representation is a series expansion in powers of  $(a_0/r)$ , where  $a_0$  is the reference equatorial radius and  $r$  is the distance from the body center of mass to the point of interest. For  $r < a_0$  the series is usually divergent; this introduces a problem for modeling dynamics within the smallest sphere that completely encloses the body. One scenario where this problem would occur would be for bodies in close proximity to an asteroid, which is generally irregular in shape.

If the shape of the body in question can be better approximated by an ellipsoid with three distinct axes, rather than a sphere or oblate spheroid, then a possible solution is to use an ellipsoidal series expansion for the potential. Ellipsoidal harmonic functions can be described as triple products of functions of whose arguments are a single ellipsoidal coordinate. Each point in space can be represented by a set of three ellipsoidal coordinates; the coordinates describe a set of surfaces which are orthogonal at that point: an ellipsoid, a hyperboloid of one sheet, and a hyperboloid of two sheets. The corresponding harmonics vanish along lines of constant values of the ellipsoidal coordinates tracing out hyperboloids of one sheet and two sheets. For the limiting case in which the ellipsoidal axes are equal, the ellipsoidal harmonics take the form of spheroidal harmonics.

This paper covers issues relevant to the implementation of ellipsoidal harmonics in existing navigation software. For example, there is not a one-to-one correspondence between Cartesian and ellipsoidal coordinates; one set of ellipsoidal coordinates can be mapped out to a point in each octant of Cartesian space. This inconvenience can be circumvented by expressing ellipsoidal harmonics in Cartesian coordinates; this formulation is more practical for implementation in existing navigation software architectures. The components of attraction are derived from the Cartesian description and are shown to converge at all points exterior to the reference ellipsoid.

At distances from the body center of mass greater than  $a(\sqrt{2}+1)$ , where  $a$  is the greatest semi-axis of the ellipsoid, an ellipsoidal harmonic of degree  $n$  can be expressed by a series of spherical harmonics of degree  $n, n-2, \dots$ . Approaches for full normalization of the coefficients are also covered for the descriptions in ellipsoidal and Cartesian coordinates. Finally, a potential field is simulated for a body whose axes have lengths resembling those of an asteroid such as Eros. Estimates are made for spheroidal and ellipsoidal model parameters. The resulting field estimates are compared to the truth model to determine the degree of field required to achieve certain levels of accuracy.