On the Semi-Definite Programming Solution of the Least Order Dynamic Output Feedback Synthesis and Related Problems

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Abstract

We show that a semi-definite programming approach can be adopted to determine the least order dynamic output feedback which stabilizes a given linear time invariant plant. The problem addressed includes as a special case, the famous static output feedback problem.

Keywords: Least Order Dynamic Output Feedback; Static Output Feedback; Semi-Definite Programming; Polynomial-time Algorithms

1 Introduction

Consider the linear time invariant (LTI) plant Σ,

$$\Sigma: \quad \dot{x} = Ax + Bu, \tag{1.1}$$

$$y = Cx, \tag{1.2}$$

where $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, and $C \in \mathbf{R}^{p\times n}$. Let $k \leq n$; represent the class of k-th order stabilizing[1] linear controllers for Σ by Σ_c^k, which have the general form,

$$\Sigma_c^k: \quad \dot{z} = A_K z + B_K y, \tag{1.3}$$

$$u = C_K z + D_K y. \tag{1.4}$$

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[1] All references to stability are in the sense of Lyapunov: the origin is the stable equilibrium point of the dynamical system $\dot{x} = Ax$ if and only if the image of a positive definite matrix under the linear map $X \rightarrow -A'X - XA$ is positive definite.

$:=$	Defined as.
$\mathbf{R}^n$	The n-dimensional Euclidean space.
$\mathbf{R}^{n\times m}$	The space of $n \times m$ matrices with entries in $\mathbf{R}$.
$\mathbf{SR}^{n\times n}$	The space of $n \times n$ symmetric matrices with entries in $\mathbf{R}$.
$\mathbf{SR}_+^{n\times n}$	The space of $n \times n$ symmetric positive semi-definite matrices with entries in $\mathbf{R}$.
A'	The transpose of the matrix A.
A^{-1}	The inverse of the matrix A.
$A^{\dagger}$	The pseudo-inverse of the matrix A.
$\mathcal{R}(A)$	The range of the matrix A.
$A > B$	The matrix difference $A - B$ is positive-definite.
$A \geq B$	The matrix difference $A - B$ is positive semi-definite (inducing the psd ordering).
$A : B$	The parallel addition of the matrices A and B.
$[\mathcal{M}]A$	The short of the matrix A over the subspace $\mathcal{M}$.

Table 1: Notation

$A_K \in \mathbf{R}^{k\times k}$.

Two major open problems in control theory are stated as follows:

1. **Static Output Feedback (SOF) Problem**: Find polynomial-time verifiable necessary and sufficient conditions on the triplet (A, B, C) such that Σ_c^0 is nonempty.[2]
2. **Least Order Dynamic Output Feedback (LODOF) Problem**: Find a polynomial time algorithm to determine the least k such that Σ_c^k is nonempty.[3]

Note that the SOF is a special case of the LODOF; we shall thus refer to both problems as the OFP (Output Feedback Problem).

The OFP has received considerable attention in systems and control community over the last thirty years [1], [9], [11], [16], [19], [20], [21], [22], [30], [31], [34], [36]; also refer to the surveys [5], [35]. In a recent survey on the state of systems and control, the OFP has been identified as an important open problem in control theory [6]. The purpose of the present paper is to "solve" the OFP using the machinery of linear matrix inequalities and semi-definite programming.

First, let us use the notation shown in Table 1, and say a few words about the problems which we shall encounter shortly.

[2]When $C = I$, a necessary and sufficient condition is stabilizability of the pair (A, B), i.e.,

$$\text{rank}\,[\lambda I - A \quad B] = n,$$

where λ ranges over the eigenvalues of A with nonnegative real parts. Similarly, when $B = I$, detectability of (C, A) provides a necessary and sufficient condition for Σ_c^0 to be nonempty.

[3]Moreover, find the corresponding k-th order controller.

The linear matrix inequality (LMI) is the problem of finding a linear combination of a given set of symmetric matrices which is positive-definite [8]. The set of all such combinations, constitute a convex set upon which a linear objective functional can be optimized. The corresponding optimization problem is referred to as semi-definite programming (SDP). Usually though the SDP is specified by the dual formulation of the problem just described, in terms of optimizing a linear functional on a convex set of symmetric matrices. The LMI and the SDP both admit efficient algorithms for their solution based on the interior-point methods. We call the LMI and the SDP polynomial-time solvable in the sense that their approximate solution can be obtained in a number of steps which is a polynomial in the size of the problem description [27].

The rank minimization problem (RMP) is the problem of finding a minimum rank matrix in a convex set of symmetric matrices, i.e.,

$$\min_{X \in \Gamma} \mathbf{rank}\ X, \tag{1.5}$$

where $\Gamma \subseteq \mathbf{SR}^{n \times n}$ for some n. The set Γ is usually described by a set of LMIs. Refer to Table 2 for a summary of these matrix optimization problems.

Some comments on the works which are directly related to the result which is presented in this paper. Motivated by the long standing difficulty in finding a polynomial-time algorithm for solving the OFP, Blondel and Tsitsiklis studied the possibility of its NP-hardness [7]. They proved for example that the SOF in particular is NP-hard, provided that one imposes bounds on the entries of the feedback gain. The polynomial-time solvability of the OFP which is proved in the present work is in the spirit of [26] and [27], where the usual notion of polynomial-time solvability of problems in discrete mathematics is modified to account for the fact that for continuous computational problems,

- finite encoding of the problem data and the candidate solution is not possible, and,
- finding an exact solution in finitely many steps is, in general, impossible.

In fact, what we demonstrate in the paper is that the OFP admits an SDP formulation.

A key step in the formulation of the OFP as an SDP is its reduction to an RMP, as demonstrated by Packard *et al.* [28], [29], El Ghaoui and Gahinet [10], and Iwasaki and Skelton [17] (also refer to [33]); we go over the proof of this result in §3.

More than a year ago, influenced by a class of problems in the complementarity theory, we realized that a class of RMPs can be solved as an SDP. This result was later submitted to the IEEE Transactions on Automatic Control. As a response to one reviewer's comments requesting for a "control application," we came across the LODOF problem and showed that the SDP formulation of the corresponding RMP gives polynomial-time computable lower and upper bounds for the least order stabilizing dynamic output feedback [23], [24]. A few technical details however, prevented us from proving that the OFP admitted a polynomial-time solution. The contribution of the present paper is to collect the necessary steps to resolve these technical issues and to show that in fact, the OFP is polynomial-time solvable. The rest of the paper is devoted to the proof of this statement which we now express as a theorem.

LMI: Given $A_i \in \mathbf{SR}^{n\times n}$ $(i = 1,\ldots,p)$, find $x \in \mathbf{R}^p$ such that,

$$\sum_{i=1}^{p} x_i A_i > 0.$$

SDP (Primal Formulation): Given $C \in \mathbf{SR}^{n\times n}$, $A_i \in \mathbf{SR}^{n\times n}$ and $b_i \in \mathbf{R}$ $(i = 1,\ldots,p)$, find $X \in \mathbf{SR}^{n\times n}$ as a solution to,

$$\begin{aligned} &\min_X \textbf{Trace}\ \ CX \\ &\textbf{Trace}\ \ A_i X = b_i \quad (1,\ldots,p), \\ &\quad X \geq 0. \end{aligned}$$

SDP (Dual Formulation): Given $c \in \mathbf{R}^p$, $A_i \in \mathbf{SR}^{n\times n}$ $(i = 1,\ldots,p)$, find $x \in \mathbf{R}^p$ as a solution to,

$$\begin{aligned} &\min_x c'x \\ &\sum_{i=1}^{p} x_i A_i > 0. \end{aligned}$$

RMP: Given a convex set $\Gamma \subseteq \mathbf{SR}^{n\times n}$, find $X \in \mathbf{SR}^{n\times n}$ as a solution to,

$$\begin{aligned} &\min_X \textbf{rank}\ X \\ &\quad X \in \Gamma. \end{aligned}$$

Table 2: LMI, SDP, and RMP

Theorem 1.1 *The OFP can be solved as an SDP. Thus the OFP is polynomial-time solvable.*[4]

2 Preliminaries

In this section we go over the concepts which are subsequently used to prove Theorem 1.1. To make the paper self-contained, a few known results are stated and proven along the way.

Given $A, B \in \mathbb{SR}^{n\times n}$, suppose that the eigenvalues of A, $\lambda_i(A)\,(i = 1,\ldots n)$, and B, $\lambda_i(B)\,(i = 1,\ldots n)$, all real, are indexed in a non-decreasing order,

$$\lambda_1(A) \leq \lambda_2(A) \leq \ldots \leq \lambda_n(A),$$
$$\lambda_1(B) \leq \lambda_2(B) \leq \ldots \leq \lambda_n(B).$$

Then [15],

$$A \leq B \;\Rightarrow\; \lambda_i(A) \leq \lambda_i(B), \quad i = 1,\ldots,n; \tag{2.6}$$

moreover, for all matrices $M \in \mathbf{R}^{p\times n}$,

$$A \leq B \;\Rightarrow\; MAM' \leq MBM', \tag{2.7}$$

implying that if M is invertible, then $MAM' \leq MBM'$ if and only if $A \leq B$.

Given $A, B \in \mathbb{SR}_+^{n\times n}$, their parallel addition $A : B$ is defined as [3],

$$A : B := A - A(A + B)^{\dagger}A,$$

and satisfies the properties,

$$0 \leq A : B \leq A, \quad 0 \leq A : B \leq B.$$

For $A \in \mathbb{SR}_+^{n\times n}$ and a subspace $\mathcal{M} \subseteq \mathbf{R}^n$, the short of A is defined as [2],

$$[\mathcal{M}]A := \max\{B \in \mathbb{SR}_+^{n\times n} \mid B \leq A, \mathcal{R}(B) \subseteq \mathcal{M}\};$$

here the max always exists [2]. For $A, B \in \mathbb{SR}_+^{n\times n}$, define the matrix interval,

$$\Delta(A, B) := \{X \mid 0 \leq X \leq A, 0 \leq X \leq B\}.$$

An extreme matrix of $\Delta(A, B)$ is a matrix $X^* \in \Delta(A, B)$ which is not majorized (with respect to the psd ordering) by any other matrix in $\Delta(A, B)$. In [4], Ando has shown that given A, B and a matrix $C \in \Delta(A, B)$, an extreme matrix of $\Delta(A, B)$, $\widetilde{X}$, can be found by letting $\mathcal{N} = \mathcal{R}(C)$,

$$Y_0 = -C + [\mathcal{N}]B, \quad \text{and} \quad Z_0 = -C + [\mathcal{N}]A,$$

[4]In the same sense that the LMI and the SDP are polynomial-time solvable [26].

and then successively iterating upon the following two equations,

$$Y_{k+1} = Y_k - Y_k : Z_k, \tag{2.8}$$
$$Z_{k+1} = Z_k - Y_k : Z_k, \tag{2.9}$$

and finally letting,

$$\tilde{X} = \frac{1}{2} \lim_{k \to \infty} \{[\mathcal{N}]A + [\mathcal{N}]B - Y_k - Z_k\}.$$

In this case we write,

$$\tilde{X} = \Delta_C^*(A, B);$$

note that $C \leq \Delta_C^*(A, B)$. Provided that the positive semi-definite matrices A, B, C above have the form,

$$A = \begin{bmatrix} A_1 & I \\ I & A_2 \end{bmatrix}, \; B = \begin{bmatrix} B_1 & I \\ I & B_2 \end{bmatrix}, \; C = \begin{bmatrix} C_1 & I \\ I & C_2 \end{bmatrix},$$

observe that,

$$C \leq \begin{bmatrix} \Delta_{C_1}^*(A_1, B_1) & I \\ I & \Delta_{C_2}^*(A_2, B_2) \end{bmatrix} \in \Delta(A, B).$$

Let Γ be a nonempty subset of $\mathbb{SR}_+^{n\times n}$. If for all $A, B \in \Gamma$, $\Delta(A, B) \cap \Gamma$ is nonempty, then Γ is called a hyper-lattice [24]. If there exists a matrix X such that $X \leq Y$ for all $Y \in \Gamma$, then we call X the (unique) least element of Γ [24].

Having stated some basic matrix theoretic facts we now recall the following results on matrix inequalities.

Proposition 2.1 *The following two statements are equivalent:*

1. $AX + XA' + Q < 0$.
2. *For all* $a > 0$, $(aI - A)X(aI - A') - (aI + A)X(aI + A') - 2aQ > 0$.

Proof: The proposition can be verified by simply expanding the left hand side of the second inequality above and dividing both sides by $a > 0$.

□

Lemma 2.2 ([12]) *Let* $M \in \mathbb{SR}^{n\times n}$, $P \in \mathbb{R}^{n\times p}$, *and* $Q \in \mathbb{R}^{q\times n}$. *Then the following statements are equivalent:*

1. *There exists a matrix* $K \in \mathbb{R}^{p\times q}$ *such that,*

$$M + PKQ + Q'K'P' < 0.$$

2. *There exists a positive number* $\bar{\gamma}$ *such that for all* $\gamma \geq \bar{\gamma}$,

$$M < \gamma PP', \quad \text{and} \quad M < \gamma Q'Q.$$

Lemma 2.3 ([28]) *Let $X \in \mathbb{SR}^{N\times N}$, X nonsingular, and $n \le N$. Let us partition X and X^{-1} as,*

$$X = \begin{bmatrix} R & \star \\ \star & \star \end{bmatrix}, \quad X^{-1} = \begin{bmatrix} S & \star \\ \star & \star \end{bmatrix},$$

where R and S are $n \times n$ symmetric matrices. Then $X > 0$ if and only if $\begin{bmatrix} R & I \\ I & S \end{bmatrix} \ge 0$ and **rank** $\begin{bmatrix} R & I \\ I & S \end{bmatrix} \le N$. *Conversely, if there are matrices $n \times n$ symmetric matrices R and S such that $\begin{bmatrix} R & I \\ I & S \end{bmatrix} \ge 0$ and* **rank** $\begin{bmatrix} R & I \\ I & S \end{bmatrix} \le N$, *then there exists a nonsingular $X \in \mathbb{SR}^{N\times N}$ such that,*

$$X = \begin{bmatrix} R & \star \\ \star & \star \end{bmatrix}, \quad X^{-1} = \begin{bmatrix} S & \star \\ \star & \star \end{bmatrix}.$$

Proposition 2.4 *Let Γ be a nonempty subset of $\mathbb{SR}_+^{n\times n}$. If Γ admits a least element, that least element has a minimal rank in Γ.*

Proof: Let X be the least element of Γ and assume that there exists $Y \in \Gamma$ such that **rank** $(Y) <$ **rank** (X), i.e., there exists an index j such that $0 = \lambda_j(Y) < \lambda_j(X)$. However, $X \le Y$, and by (2.6), $\lambda_j(X) \le \lambda_j(Y)$, thus establishing a contradiction.

□

Lastly, we state the Kakutani's fixed point theorem and a related definition for completeness.

Definition 2.1 *Let S be a closed bounded convex set in an Euclidean space and $R(S)$ be the family of all closed convex subsets of S. A point-to-set mapping $x \to f(x)$ from S into $R(S)$ is called upper semi-continuous if $x_n \to x^*$, $y_n \in f(x_n)$, and $y_n \to y^*$, imply that $y^* \in f(x^*)$.*

Theorem 2.5 ([18]) *If $x \to f(x)$ is an upper semi-continuous point-to-set mapping of a bounded closed convex set S in an Euclidean space into $R(S)$, then there exists $\bar{x} \in S$ such that $\bar{x} \in f(\bar{x})$.*

3 Proof of Theorem 1.1

The proof of the theorem proceeds along the following lines. In §3.1 it is first shown that an RMP of the form (1.5) whose feasible set is a hyper-lattice can be solved as an SDP (Lemma 3.1). The OFP is then shown to be equivalent to an RMP (Lemma 3.2) (the main part of this result has been proven (or stated) in [10], and [17], and [29]). After stating some related technical issues in §3.2, we proceed to prove that the feasible set of the corresponding RMP is a hyper-lattice in §3.3 (Lemma 3.8), thus completing the proof of the theorem. Some of the related technical issues are gathered in terms of various propositions in §3.2.

3.1

In subsequent sections we assume that the feasible sets of the SDPs or the RMPs are nonempty; note that the feasibility of an SDP, or an RMP whose feasible set is defined by a set of LMIs, can always be checked via the interior point method in polynomial-time.

Lemma 3.1 *Let $\Gamma \subseteq \mathbf{SR}_+^{n\times n}$ be nonempty and compact. If Γ is a hyper-lattice then,*

$$X^* := \arg\min_{X\in\Gamma} \mathbf{Trace}\ X,$$

is of minimal rank in Γ.[5]

Proof: The matrix X^* exists by the compactness of Γ and continuity of the trace functional. Let $Y \in \Gamma$ be arbitrary, and $Z \in \Delta(X^*, Y)$. By optimality of X^*, $\mathbf{Trace}\ (X^* - Z) \leq 0$. By the choice of Z however, $\mathbf{Trace}\ (Z - X^*) \leq 0$, and thus $\mathbf{Trace}\ (X^* - Z) = 0$. Since $X^* - Z \geq 0$ and $\mathbf{Trace}\ (X^* - Z) = 0$, $Z = X^*$. Thereby, $X^* \leq Y$, for all $Y \in \Gamma$, and X^* is the least element of Γ. In view of Proposition 2.4, X^* has minimum rank in Γ.[6] □

Lemma 3.2 ([10], [17]) *There exists $\bar{\gamma} > 0$ such that for every $\gamma \geq \bar{\gamma}$, the OFP can be written as the following optimization problem,*

$$\min_{X,Y} \mathbf{rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \tag{3.10}$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0, \tag{3.11}$$

$$AX + XA' < \gamma BB', \tag{3.12}$$

$$A'Y + YA < \gamma C'C, \tag{3.13}$$

Proof: Combining the dynamics of the plant (1.1)-(1.2) and that of the controller (1.3)-(1.4), one obtains,

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} &= \begin{bmatrix} A + BD_K C & BC_K \\ B_K C & A_K \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \\ &= (\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & B \\ I & 0 \end{bmatrix} \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \begin{bmatrix} 0 & I \\ C & 0 \end{bmatrix}) \begin{bmatrix} x \\ z \end{bmatrix} \\ &= (\tilde{A} + \tilde{B}K\tilde{C}) \begin{bmatrix} x \\ z \end{bmatrix}, \end{aligned} \tag{3.14}$$

where,

$$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \tilde{B} = \begin{bmatrix} 0 & B \\ I & 0 \end{bmatrix}, \tilde{C} = \begin{bmatrix} 0 & I \\ C & 0 \end{bmatrix}, K = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix};$$

[5] In fact, the proof establishes a stronger statement: every nonempty compact hyper-lattice admits a least element which can be found by minimizing the trace functional over it.

[6] When Γ is a set defined by LMIs and the least element (and thus the minimum rank solution) is found approximately using a numerical method, care must be given in the determination of the rank of the corresponding approximate least element.

$\widetilde{A} \in \mathbf{R}^{(n+k)\times(n+k)}$. Note how (3.14) reduces to the static output feedback case when $k = 0$.

Now, according to the Lyapunov's stability criterion, the origin is the stable equilibrium point of the the closed loop system (3.14) if and only if the following matrix inequalities are feasible,

$$\begin{aligned} &(\widetilde{A} + \widetilde{B}K\widetilde{C})\widetilde{X} + \widetilde{X}(\widetilde{A} + \widetilde{B}K\widetilde{C})' \\ &= (\widetilde{A}\widetilde{X} + \widetilde{X}\widetilde{A}') + (\widetilde{B}K\widetilde{C}\widetilde{X}) + (\widetilde{X}\widetilde{C}'K'\widetilde{B}') < 0, \qquad (3.15)\\ &\widetilde{X} > 0. \qquad (3.16)\end{aligned}$$

In view of Lemma 2.2, (3.15)-(3.16) is equivalent to the existence of $\gamma > 0$, such that the following matrix inequalities are feasible,

$$\begin{aligned} \widetilde{A}\widetilde{X} + \widetilde{X}\widetilde{A}' &< \gamma\widetilde{B}\widetilde{B}', \\ \widetilde{A}\widetilde{X} + \widetilde{X}\widetilde{A}' &< \gamma\widetilde{X}\widetilde{C}'\widetilde{C}\widetilde{X}, \\ \widetilde{X} &> 0. \end{aligned}$$

Note that these inequalities are nonconvex in $\widetilde{X}$; in fact they are in the form of a bilinear matrix inequality (BMI) [25], [32]. Letting $\widetilde{Y} = \widetilde{X}^{-1}$ and using (2.7), we conclude that (3.15)-(3.16) is equivalent to,

$$\widetilde{A}\widetilde{X} + \widetilde{X}\widetilde{A}' < \gamma\widetilde{B}\widetilde{B}', \qquad (3.17)$$

$$\widetilde{A}'\widetilde{Y} + \widetilde{Y}\widetilde{A} < \gamma\widetilde{C}'\widetilde{C}, \qquad (3.18)$$

$$\widetilde{X} > 0, \qquad (3.19)$$

$$\widetilde{X}\widetilde{Y} = I. \qquad (3.20)$$

We notice that the LMIs (3.17)-(3.19), with an additional non-convex constraint (3.20), are in terms of the matrices $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{C}$. Since these matrices contain zero blocks, (3.17)-(3.20) can be simplified and rewritten in terms of the original triplet (A, B, C) as we now proceed to show.

Let $\widetilde{X} = \begin{bmatrix} X & \star \\ \star & \star \end{bmatrix}$ and $\widetilde{Y} = \begin{bmatrix} Y & \star \\ \star & \star \end{bmatrix}$. If (3.17)-(3.19) hold, then a simple block matrix multiplication shows that,

$$AX + XA' < \gamma BB', \qquad (3.21)$$

$$A'Y + YA < \gamma C'C, \qquad (3.22)$$

$$X > 0, \qquad (3.23)$$

and according to Lemma 2.3, (3.20) implies,

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0, \ \mathbf{rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \leq n + k. \qquad (3.24)$$

Thus the feasibility of (3.17)-(3.20) implies that of (3.21)-(3.24).

On the other hand, (3.24) and Lemma 2.3 imply the existence of the positive definite matrices $\tilde{X} = \begin{bmatrix} X & M \\ M' & \star \end{bmatrix}$ and $\tilde{Y} = \begin{bmatrix} Y & N \\ N' & \star \end{bmatrix}$, such that $\tilde{X}\tilde{Y} = I$. Moreover since,

$$\tilde{A}\tilde{X} + \tilde{X}\tilde{A}' - \gamma_1 \tilde{B}\tilde{B}' = \begin{bmatrix} AX + XA' - \gamma_1 BB' & AM' \\ MA' & -\gamma_1 I \end{bmatrix},$$

(3.17) holds for some $\gamma_1 > 0$. Similarly (3.18) holds by invoking (3.22) and Lemma 2.3, and choosing the appropriate $\gamma_2 > 0$. It now suffices to let γ be equal to $\max\{\gamma_1, \gamma_2\}$. Thus the feasibility of (3.21)-(3.24) implies that of (3.17)-(3.20) for some $\gamma > 0$. Consequently (3.17)-(3.20) and (3.21)-(3.24) are equivalent for an appropriately chosen $\gamma > 0$. □

Proposition 3.3 *Given the matrices X and Y as solutions to the optimization problem (3.10)-(3.13), the corresponding stabilizing static or least order dynamic output feedback controllers can be found using an LMI.*

Proof: Note that solving (3.15)-(3.16) for $\tilde{X}$ and K is equivalent to the OFP. Knowing $\tilde{X}$ however, reduces (3.15)-(3.16) to an LMI. It is thus sufficient to show that $\tilde{X}$ can be constructed from the optimal X and Y of (3.10)-(3.13). The construction plan is implicit in Lemma 2.3: given X and Y, $\tilde{X}$ is such that,

$$\tilde{X} = \begin{bmatrix} X & \tilde{X}_{12} \\ \tilde{X}_{12} & \tilde{X}_{22} \end{bmatrix}, \quad \text{and} \quad \tilde{X}^{-1} = \begin{bmatrix} Y & \star \\ \star & \star \end{bmatrix};$$

thereby,

$$X - Y^{-1} = \tilde{X}_{12}\tilde{X}_{22}^{-1}\tilde{X}_{12}.$$

Since $\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0$, $X - Y^{-1} \geq 0$. Let $\begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ be the singular value decomposition of $X - Y^{-1}$, with U_1 and U_2 symmetric, and S diagonal with positive entries; set,

$$\tilde{X} = \begin{bmatrix} X & U_1 \\ U_1 & S^{-1} \end{bmatrix}.$$

□

3.2

In this section, we gather few technical issues related to the final proof of Theorem 1.1 in §3.3. These results are concerned about establishing that the feasible set of the RMP which corresponds to the OFP can be represented as a hyper-lattice.

Proposition 3.4 *There exist $q > 0$ and $\bar{\gamma} > 0$ such that for every $\gamma \geq \bar{\gamma}$ the OFP can be written as the following optimization problem,*

$$\min_{X,Y} \textbf{rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \tag{3.25}$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0, \tag{3.26}$$

$$AX + XA' < \gamma BB', \tag{3.27}$$

$$A'Y + YA < \gamma C'C, \tag{3.28}$$

$$qI \geq \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0. \tag{3.29}$$

Proof: Given the feasibility of (3.11)-(3.13), the existence of the positive number q is clear.

□

Proposition 3.5 *There exist $q > 0$, $\epsilon > 0$, and $\bar{\gamma} > 0$, such that for every $\gamma \geq \bar{\gamma}$ the OFP can be written as the following optimization problem,*

$$\min_{Z} \textbf{rank } Z \tag{3.30}$$

$$Z - NZN' + Q_\gamma \geq 0, \tag{3.31}$$

$$Z = \begin{bmatrix} Z_1 & I \\ I & Z_2 \end{bmatrix}, \tag{3.32}$$

$$qI \geq Z \geq 0, \tag{3.33}$$

where,

$$J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \tag{3.34}$$

$$N = \begin{bmatrix} (aI - A)^{-1}(aI + A) & 0 \\ 0 & (bI - A')^{-1}(bI + A') \end{bmatrix}, \tag{3.35}$$

$$Q_\gamma = 2\gamma \begin{bmatrix} a(aI - A)^{-1}BB'(aI - A)' & 0 \\ 0 & b(bI - A)^{-1}C'C(bI - A')^{-1} \end{bmatrix} \tag{3.36}$$

$$- \quad J + NJN' - \epsilon I. \tag{3.37}$$

Proof: According to Proposition 2.1, for any $\alpha > 0$, the inequality (3.27) can be written as,

$$(aI - A)X(aI - A)' - (aI + A)X(aI + A)' + 2a\gamma BB' > 0. \tag{3.38}$$

Choose $a > 0$ such that $aI - A$ is nonsingular and define,

$$N_1 = aI - A, \quad \text{and} \quad M_1 = aI + A.$$

Thus (3.38) is equivalent to,

$$X - N_a X N_a' + 2a\gamma B_a B_a' > 0, \tag{3.39}$$

where,

$$N_a = N_1^{-1} M_1, \quad B_a = N_1^{-1} B.$$

Similarly (3.28) can be written as,

$$Y - N_b Y N_b' + 2b\gamma C_b' C_b > 0, \tag{3.40}$$

where,

$$N_2 = bI - A', \qquad M_2 = bI + A',$$

and,

$$N_b = N_2^{-1} M_2, \quad C_b = C N_2^{-1};$$

the number b is chosen such that $bI - A'$ is nonsingular. Putting (3.39) and (3.40) together we obtain,

$$\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} - \begin{bmatrix} N_a & 0 \\ 0 & N_b \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \begin{bmatrix} N_a & 0 \\ 0 & N_b \end{bmatrix}' + 2\gamma \begin{bmatrix} aB_a B_a' & 0 \\ 0 & bC_b' C_b \end{bmatrix} > 0. \tag{3.41}$$

Let,

$$N := \begin{bmatrix} N_a & 0 \\ 0 & N_b \end{bmatrix}, \tag{3.42}$$

$$J := \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \tag{3.43}$$

$$Q_0 := \begin{bmatrix} aB_a B_a' & 0 \\ 0 & bC_b' C_b \end{bmatrix}. \tag{3.44}$$

Rewriting (3.25)-(3.29) by letting $Z = \begin{bmatrix} X & I \\ I & Y \end{bmatrix}$, one obtains the following RMP,

$$\begin{aligned} &\min_Z \textbf{rank } Z \\ Z - NZN' + (2\gamma Q_0 - J + NJN' - \epsilon I) &\geq 0, \\ Z &= \begin{bmatrix} Z_1 & I \\ I & Z_2 \end{bmatrix}, \\ qI \geq Z &\geq 0, \end{aligned}$$

for some $\epsilon > 0$ and $q > 0$. Note that the matrices N and Q_0 are block diagonal. Let,

$$Q_\gamma := 2\gamma Q_0 - J + NJN' - \epsilon I. \tag{3.45}$$

□

Proposition 3.6 *Let the matrices N and Q_γ be defined as in (3.35) and (3.37). Then for all matrices X of the form,*

$$X = \begin{bmatrix} X_1 & I \\ I & X_2 \end{bmatrix} \geq 0$$

and every $\epsilon > 0$, there exist positive scalars a, b, and symmetric matrices U_1 and U_2, such that,

$$NXN' - Q_\gamma \geq \begin{bmatrix} U_1 & I \\ I & U_2 \end{bmatrix} \geq 0. \tag{3.46}$$

Proof: The quadratic term in (a, b) in the inequality (3.46) is of the form,

$$\begin{bmatrix} aI & 0 \\ 0 & bI \end{bmatrix} \begin{bmatrix} X_1 + \epsilon I & I \\ I & X_2 + \epsilon I \end{bmatrix} \begin{bmatrix} aI & 0 \\ 0 & bI \end{bmatrix} + \text{linear terms in } a \text{ and } b.$$

Thus if $\begin{bmatrix} X_1 & I \\ I & X_2 \end{bmatrix} \geq 0$, for every $\epsilon > 0$, there exist $a, b > 0$ such that (3.46) holds for some matrix of the form $\begin{bmatrix} U_1 & I \\ I & U_2 \end{bmatrix}$. □

Corollary 3.7 *There exist positive scalars a, b, q, ϵ, $\bar{\gamma} > 0$, and symmetric matrices U_1, U_2, such that for every $\gamma \geq \bar{\gamma}$, the OFP can be written as the following optimization problem,*

$$\min_Z \textbf{rank } Z \tag{3.47}$$

$$Z - NZN' + Q_\gamma \geq 0, \tag{3.48}$$

$$Z = \begin{bmatrix} Z_1 & I \\ I & Z_2 \end{bmatrix}, \tag{3.49}$$

$$qI \geq Z \geq \begin{bmatrix} U_1 & I \\ I & U_2 \end{bmatrix} \geq 0, \tag{3.50}$$

where the matrices N and Q_γ are as defined in (3.35) and (3.37), respectively.

Proof: In view of Proposition 3.6, note that $a, b > 0$ and symmetric matrices can be chosen such that, for every Z of the form $\begin{bmatrix} \star & I \\ I & \star \end{bmatrix} \geq 0$,

$$NZN' - Q_\gamma \geq \begin{bmatrix} U_1 & I \\ I & U_2 \end{bmatrix} \geq 0. \tag{3.51}$$

Given that for a feasible Z, $Z \geq NZN' - Q_\gamma$, (3.50) is automatically satisfied. □

3.3

With matrices N (3.35), Q_γ (3.37), fixed $\gamma > 0$, and $U = \begin{bmatrix} U_1 & I \\ I & U_2 \end{bmatrix}$ as in Corollary 3.7, define the set,

$$\Gamma := \{X \mid qI \geq X = \begin{bmatrix} X_1 & I \\ I & X_2 \end{bmatrix} \geq U,\ X - NXN' + Q_\gamma \geq 0\}. \tag{3.52}$$

The following lemma connects together all the results which we have developed so far for the solution of the OFP.

Lemma 3.8 ***For a fixed $\gamma > 0$, and suitable choices of a and b, Γ (3.52) is a hyper-lattice.***

Proof: **Let $X, Y \in \Gamma$; define the set,**

$$\bar{\Delta}(X,Y) := \{Z \in \Delta(X,Y) \mid Z = \begin{bmatrix} Z_1 & I \\ I & Z_2 \end{bmatrix}\} \subseteq \Delta(X,Y).$$

Note that $\bar{\Delta}(X,Y)$ is nonempty (in contains U).

For all $Z \in \bar{\Delta}(X,Y)$, and appropriately chosen a and b (refer to Proposition 3.6) one has,

$$X \geq NXN' - Q_\gamma \geq NZN' - Q_\gamma \geq 0,$$

and,

$$Y \geq NYN' - Q_\gamma \geq NZN' - Q_\gamma \geq 0.$$

By the structure of the N we conclude that,

$$NZN' - Q_\gamma = \begin{bmatrix} \tilde{Z}_1 & I \\ I & \tilde{Z}_2 \end{bmatrix},$$

for some matrices $\tilde{Z}_1$ and $\tilde{Z}_2$, since,

$$\begin{aligned} NZN' - Q_\gamma &= NZ_0N' + NJN' - (2\gamma Q_0 - J + NJN' - \epsilon I) \\ &= NZ_0N' + J - 2\gamma Q_0 + \epsilon I. \end{aligned}$$

Let,

$$W := \begin{bmatrix} \Delta^*_{\tilde{Z}_1}(X_1, Y_1) & I \\ I & \Delta^*_{\tilde{Z}_2}(X_2, Y_2) \end{bmatrix} \in \Delta_{NZN'-Q}(X,Y). \tag{3.53}$$

Note that $W \in \bar{\Delta}(X,Y)$ (refer to §2). Thus, for every $Z \in \bar{\Delta}(X,Y)$, there exists $W = \begin{bmatrix} W_1 & I \\ I & W_2 \end{bmatrix} \in \bar{\Delta}(X,Y)$ such that,

$$W \geq NZN' - Q_\gamma;$$

express such an assignment by the map $f : \bar{\Delta}(X,Y) \rightarrow \bar{\Delta}(X,Y)$. Observe that f is upper semi-continuous: let $\{Z_k\}_{k\geq 1}$ and $\{W_k\}_{k\geq 1}$ be a sequence of matrices such that,

$$W_k \geq N Z_k N' - Q_\gamma,$$

and let $Z_k \rightarrow Z^*$, and $W_k \rightarrow W^*$. Define the map,

$$M(Z_k, W_k) := W_k - N Z_k N' + Q_\gamma.$$

The map M is linear on $\mathbb{SR}^{n\times n} \times \mathbb{SR}^{n\times n}$ and thereby continuous. Since the cone of positive semi-definite matrices is closed,

$$0 \leq \lim_{k\rightarrow\infty} M(Z_k, W_k) = M(Z^*, W^*),$$

and,

$$W^* \geq N Z^* N' - Q_\gamma;$$

hence $W^* \in f(Z^*)$.

Note also that $\bar{\Delta}(X,Y)$ is a bounded closed convex set. Consequently by the Kakutani's Fixed Point Theorem (Theorem 2.5), there exists a matrix $\bar{Z} \in \bar{\Delta}(X,Y)$ such that,

$$\bar{Z} = f(\bar{Z}),$$

i.e.,

$$\bar{Z} \geq N \bar{Z} N' - Q_\gamma.$$

This implies that for all $X, Y \in \Gamma$, $\Gamma \cap \Delta(X,Y)$ is nonempty, proving that the set Γ is in fact a hyper-lattice for the suitable choices of the scalars $a, b > 0$.

$\square$

We have thus proved the following statement: for appropriate scalars $a, b > 0$, the feasible set of the RMP which corresponds to the OFP is a hyper-lattice. Consequently, according to Lemma 3.1, the OFP can be solved as an SDP. The SDP formulation of the OFP is of the form,

$$\min_{X,\gamma>0} \text{Trace } X \tag{3.54}$$

$$X - NXN' + Q_\gamma \geq 0, \tag{3.55}$$

$$X = \begin{bmatrix} X_1 & I \\ I & X_2 \end{bmatrix}, \tag{3.56}$$

$$X \geq 0; \tag{3.57}$$

note once again that the parameters $a, b > 0$ are hidden inside the matrix N (3.35). The parameter $\gamma > 0$ in Q_γ (3.37), on the other hand, can be iterated upon or fixed to some large positive value.

Rewriting the above SDP in terms of the original triplet (A, B, C), one obtains,

$$\min_{X,Y} \textbf{Trace } X + Y \tag{3.58}$$

$$XA' + AX \leq \gamma BB' + \frac{-\epsilon}{2a}(a^2 I - aA' - aA + AA'), \tag{3.59}$$

$$A'Y + YA \leq \gamma C'C + \frac{-\epsilon}{2b}(b^2 I - bA - bA' + AA'), \tag{3.60}$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0. \tag{3.61}$$

What we have proved in the paper is that for a suitable choice of a and b, the above SDP is equivalent to OFP.

We demonstrate the applicability of the proposed SDP approach to the OFP via some examples.

4 Examples

We used the LMITOOL, developed by El Ghaoui, Delebecque, and Nikoukhah [13] (an interface to the SP Package of Vandenberghe and Boyd), to solve the SDP formulation of the OFP proposed in the paper.[7]

Our first example is the 2-mass spring system as considered for example in [14], where two bodies with equal unit mass are connected by a spring with unit stiffness. It is assumed that the problem is non–collocated, i.e., that the control force acts on one body and the position is measured on the other. The system can be described in the form of (1.1)-(1.2) with,

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}.$$

It is known that a second order controller is a minimum order dynamic output feedback which stabilizes this system. We formulated this LODOF problem as an SDP of the form (3.54)-(3.57) by setting $a = b = 10^6$, and $\epsilon = 10^{-7}$. The solution of the corresponding RMP is found to be,

$$X^* = \begin{bmatrix} 0.9277 & 0.5462 & -0.0010 & -0.0094 & 1.0000 & 0 & 0 & 0 \\ 0.5462 & 1.8685 & 0.0042 & -0.0091 & 0 & 1.0000 & 0 & 0 \\ -0.0010 & 0.0042 & 0.6532 & -0.3815 & 0 & 0 & 1.0000 & 0 \\ -0.0094 & -0.0091 & -0.3815 & 1.3224 & 0 & 0 & 0 & 1.000 \\ 1.0000 & 0 & 0 & 0 & 1.3224 & -0.3815 & -0.0091 & -0.0094 \\ 0 & 1.0000 & 0 & 0 & -0.3815 & 0.6532 & 0.0042 & -0.0010 \\ 0 & 0 & 1.0000 & 0 & -0.0091 & 0.0042 & 1.8685 & 0.5462 \\ 0 & 0 & 0 & 1.0000 & -0.0094 & -0.0010 & 0.5462 & 0.9277 \end{bmatrix};$$

[7] We have recently noted that, in some cases, a non-interior point based algorithms also performs well for solving the SDP formulation of the OFP.

the associated vector of eigenvalues is,

$$\Lambda = \begin{bmatrix} 0.0000 \\ 0.0000 \\ 0.0083 \\ 0.0172 \\ 2.1692 \\ 2.1635 \\ 2.5911 \\ 2.5943 \end{bmatrix};$$

we conclude that the minimum order stabilizing dynamic output feedback for the two mass-spring system is in fact two.[8]

Our second example is based on a random selection of the triplet (A, B, C) in (1.1)-(1.2):

$$A = \begin{bmatrix} -0.9015 & -1.9020 & 1.2065 & 0.4029 & 0.5304 & -2.1496 & -0.7712 & -2.0484 \\ 5.2170 & -2.7524 & 2.1887 & 1.6255 & 3.6976 & -7.5229 & 2.3800 & -2.5410 \\ -3.5108 & 2.3465 & -1.2270 & -1.5700 & -2.4693 & 3.7815 & -0.8531 & 1.4200 \\ -4.0187 & 1.7917 & -3.7214 & -4.4778 & -3.8795 & 2.7415 & 1.1050 & 2.0385 \\ 0.1723 & -2.0873 & -2.2474 & -2.0884 & -0.5313 & -1.8846 & 2.7407 & -0.8183 \\ 0.6402 & 1.2032 & 0.7113 & -0.6443 & 1.0924 & -3.5431 & 0.4691 & 1.7167 \\ 3.3423 & -1.3419 & -2.1090 & -1.7480 & -0.8293 & -1.7091 & -0.4512 & -1.6132 \\ -2.3254 & -0.8906 & -1.9334 & -2.3926 & -2.7666 & 0.8354 & 1.8646 & -2.2624 \end{bmatrix},$$

$$B = \begin{bmatrix} -0.4051 & -1.1717 & -0.9020 \\ 0.2923 & 2.0329 & -2.0533 \\ 2.5659 & 0.9685 & 0.0891 \\ -0.4578 & 0.6703 & 2.0871 \\ -1.6108 & 0.4201 & 0.3651 \\ -2.6695 & -2.8728 & 0.8461 \\ -0.7597 & 1.6859 & -0.1845 \\ -0.6747 & 0.0279 & 1.0307 \end{bmatrix}, \quad \text{and,}$$

$$C = \begin{bmatrix} -1.5276 & 0.5262 & 0.1988 & 0.0322 & -1.2992 & 1.8175 & -1.0107 & 0.6912 \\ 0.9649 & -0.1845 & 1.5904 & 0.8892 & 1.1826 & -0.5843 & -0.9605 & -0.7586 \end{bmatrix}.$$

We note that the minimum order stabilizing dynamic output feedback for a generic system is zero, i.e., for random choices of A, B, and C, static output feedback can be used to stabilize the system. The solution of the corresponding RMP was found using

[8] Since $n = 4$ and $n + k = 6$.

the SDP formulation (3.54)-(3.57) with the following vector of eigenvalues,[9]

$$\Lambda = \begin{bmatrix} 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ -0.0000 \\ 0.0000 \\ 2.0000 \\ 2.0000 \\ 2.0000 \\ 2.0000 \\ 2.0000 \\ 2.0000 \\ 2.1786 \\ 2.1758 \end{bmatrix}.$$

Thus, a static output feedback can be used to stabilize this system.[10]

Note that in view of Proposition 3.3, having the pair (X, Y) (the upper left and lower right submatrices of X^*), the corresponding controller for both examples above can be found via solving the LMI (3.15).

5 Concluding Remarks

It is shown that the RMP resulting from the least order dynamic output feedback can be solved as a semi-definite program. As an immediate consequence of this result, it is concluded that the OFP is polynomial time solvable, thus settling two famous open problems in control theory.

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[9] $X^* \in \mathbb{SR}^{16\times 16}$.

[10] Since $n = 8$ and $n + k = 8$.

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