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Sensitivities of alternate LISA configurations

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Abstract
LISA can be considered as six proof masses, pairwise Doppler tracked with one-way optical links. An appropriate combination of these one-way Doppler links can cancel the principal noise sources (laser phase noise and optical bench motion noise). Here we compute the gravitational wave sensitivities for the baseline Michelson combination and for several other LISA data combinations which also cancel the leading noise.

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1. Introduction

This paper describes a calculation of LISA's sensitivity for several laser-phase-noise cancelling configurations. The procedure has been called time-delay interferometry [1–3]. We use the notation and conventions of [3]. The spacecraft (labelled 1, 2, 3) are equidistant (distance, l) from point O. Relative to O, the spacecraft are located by coplanar unit vectors $\hat{p}_1$, $\hat{p}_2$, $\hat{p}_3$. The lengths between pairs of spacecraft are L_1, L_2, L_3, with L_i being opposite spacecraft i. Unit vectors along lines connecting spacecraft pairs, $\hat{n}_i$, are oriented with $\hat{n}_1$'s foot at spacecraft 3 and arrow toward spacecraft 2, $\hat{n}_3$'s foot at spacecraft 2 and arrow toward spacecraft 1 and $\hat{n}_2$'s foot at spacecraft 1 and arrow toward spacecraft 3.

The laser beams exchanged between spacecraft pairs, y_{ij}, are labelled by the spacecraft transmitting and receiving the beam. The convention is that y_{12} is the beam received at spacecraft 2 and transmitted from spacecraft 3, y_{13} is the beam received at spacecraft 3 and transmitted from spacecraft 2, etc. Internal metrology data to correct for optical bench motions are denoted z_{ij}. (The information content and labelling convention of the z_{ij} are described in [3].) Delay of laser data streams, either by time of flight or in signal processing, is indicated by commas in the y_{ij} subscripts: $y_{31,2} = y_{31}(t - L_2)$, etc $c = 1$ is used in the formulation; conversion to physical units is done for the results. The proof-mass-plus-optical-bench assemblies for LISA spacecraft have been described elsewhere [3, 4].

2. Signal and noise transfer functions in a single laser link

The response of the one-way Doppler time series y_{31} and y_{21} excited by a transverse, traceless plane gravitational wave having unit wavevector $\hat{k}$ is, in the above notation [5–7]:

$$y_{31}^{gw}(t) = \left[1 + \frac{l}{L_3}(\mu_1 - \mu_2)\right](\Psi_3(t - \mu_2 l - L_3) - \Psi_3(t - \mu_1 l)) \tag{1}$$

$$y_{21}^{gw}(t) = \left[1 - \frac{l}{L_2}(\mu_3 - \mu_1)\right](\Psi_2(t - \mu_3 l - L_2) - \Psi_2(t - \mu_1 l)) \tag{2}$$

where $\mu_i = \hat{k} \cdot \hat{p}_i$, Ψ_i is

$$\Psi_i(t) = \frac{1}{2}\frac{\hat{n}_i \cdot \boldsymbol{h}(t) \cdot \hat{n}_i}{1 - (\hat{k} \cdot \hat{n}_i)^2} \tag{3}$$

and $\boldsymbol{h}(t)$ is the metric perturbation at point O. Note that $L_1 \hat{k} \cdot \hat{n}_1 = l(\mu_2 - \mu_3)$, and so forth by cyclic index permutation. The gravitational wave $\boldsymbol{h}(t)$ is $\left[h_+(t)\boldsymbol{e}_+ + h_\times(t)\boldsymbol{e}_\times\right]$, where the 3-tensors $\boldsymbol{e}_+$ and $\boldsymbol{e}_\times$ are transverse to $\hat{k}$ and traceless. The GW contribution of the other four y_{ij}s can be obtained by cyclic index permutation. These each have a two-pulse response: a δ-functions incident wave is replicated at two different times in each y_{ij}. The noise contributions to y_{ij} and z_{ij} Doppler measurements were developed in [3] (equations (2.1)–(2.4)).

3. Noise-cancelling data combinations

Assuming a rigid LISA configuration with all lasers having the same centre frequency, combinations of the y_{ij} data which exactly cancel laser and optical-bench-motion noise have been derived [3]. These combinations have residual proof mass motion noise and shot noise. In this section we simply restate [3] the results required to perform sensitivity calculations for conventional and alternate LISA noise-cancelling configurations.

There are four noise-cancelling data combinations which involve only four of the y_{ij}: the interferometer (X, Y, X), beacon (P, Q, R), monitor (E, F, G) and relay (U, V, W) combinations. These all have 'eight-pulse' responses in that an impulsive GW is copied eight times, with amplitudes and time-offsets depending on wave properties, into the noise-cancelling combination.

The nominal LISA data combination is an unequal-arm Michelson interferometer. The appropriate time-domain combination uses the y_{ij} from only two arms and from their intra-spacecraft z_{ij} to cancel laser and optical bench noise [1]. There are three possible synthesized interferometers: $X = y_{32,322} - y_{23,233} + y_{31,22} - y_{21,33} + y_{23,2} - y_{32,3} + y_{21} - y_{31} + \frac{1}{2}(-z_{21,2233} + z_{21,33} + z_{21,22} - z_{21}) + \frac{1}{2}(+z_{31,2233} - z_{31,33} - z_{31,22} + z_{31})$, with Y and Z given by cyclic index permutation. The GW response in X is explicitly

$$\begin{aligned}
X^{gw} = {} & \left[1 - \frac{l}{L_3}(\mu_1 - \mu_2)\right](\Psi_3(t - \mu_1 l - 2L_3 - 2L_2) - \Psi_3(t - \mu_2 l - L_3 - 2L_2)) \\
& - \left[1 + \frac{l}{L_2}(\mu_3 - \mu_1)\right](\Psi_2(t - \mu_1 l - 2L_2 - 2L_3) \\
& - \Psi_2(t - \mu_3 l - L_2 - 2L_3)) \\
& + \left[1 + \frac{l}{L_3}(\mu_1 - \mu_2)\right](\Psi_3(t - \mu_2 l - L_3 - 2L_2) - \Psi_3(t - \mu_1 l - 2L_2))
\end{aligned}$$

$$-\left[1-\frac{l}{L_2}(\mu_3-\mu_1)\right](\Psi_2(t-\mu_3 l-L_2-2L_3)-\Psi_2(t-\mu_1 l-2L_3))$$

$$+\left[1+\frac{l}{L_2}(\mu_3-\mu_1)\right](\Psi_2(t-\mu_1 l-2L_2)-\Psi_2(t-\mu_3 l-L_2))$$

$$-\left[1-\frac{l}{L_3}(\mu_1-\mu_2)\right](\Psi_3(t-\mu_1 l-2L_3)-\Psi_3(t-\mu_2 l-L_3))$$

$$+\left[1-\frac{l}{L_2}(\mu_3-\mu_1)\right](\Psi_2(t-\mu_3 l-L_2)-\Psi_2(t-\mu_1 l))$$

$$-\left[1+\frac{l}{L_3}(\mu_1-\mu_2)\right](\Psi_3(t-\mu_2 l-L_3)-\Psi_3(t-\mu_1 l)) \qquad (4)$$

with Ψ_i and μ_i given in terms of the wave properties and detector geometry as above. The power spectra of acceleration and shot noise in X, assuming independent and equal individual proof mass acceleration noise and independent and equal shot noise for the equilateral triangle ($L_i = L$) case, enter X as $S_X = [8\sin^2(4\pi f L) + 32\sin^2(2\pi f L)]S_y^{\textit{proof mass}} + 16\sin^2(2\pi f L)\, S_y^{\textit{optical path}}$. (In the above 'shot noise' and 'optical path noise' have been used synonymously.)

In addition to the unequal-arm-length interferometer configurations, the (P, Q, R), (E, F, G) and (U, V, W) combinations also cancel laser and optical bench noise. $P = y_{32,2} - y_{23,3} - y_{12,2} + y_{13,3} + y_{12,13} - y_{13,12} + y_{23,311} - y_{32,211} + \frac{1}{2}(-z_{21,23} + z_{21,1123} + z_{31,23} - z_{31,1123}) + \frac{1}{2}(-z_{32,2} + z_{32,112} + z_{12,2} - z_{12,112}) + \frac{1}{2}(-z_{13,3} + z_{13,113} + z_{23,3} - z_{23,113})$, with Q and R given by index permutation. The noise power spectrum for P is $S_P = [8\sin^2(2\pi f L) + 32\sin^2(\pi f L)]S_y^{\textit{proof mass}} + [8\sin^2(2\pi f L) + 8\sin^2(\pi f L)]S_y^{\textit{optical path}}$.

The monitor combination is $E = y_{12,21} - y_{13,31} - y_{12,3} + y_{13,2} + y_{31,11} - y_{21,11} - y_{31} + y_{21} - \frac{1}{2}(z_{13,2} + z_{21} + z_{32,3} - z_{13,112} + z_{23,112} - z_{32,113}) + \frac{1}{2}(z_{23,2} + z_{31} + z_{12,3} - z_{12,113} + z_{21,11} - z_{31,11})$. The gravitational wave response of E is implicit in the relation $E = \alpha - \zeta_{,1}$. The noise power spectrum in E is $S_E = [32\sin^2(\pi f L) + 8\sin^2(2\pi f L)]S_y^{\textit{proof mass}} + [8\sin^2(\pi f L) + 8\sin^2(2\pi f L)]S_y^{\textit{optical path}}$. Combinations F and G are obtained from cyclic index permutation.

The relay combination is $U = y_{21,113} - y_{21,3} - y_{12,123} + y_{13,1} - y_{13,23} + y_{32,11} - y_{32} + y_{12} - \frac{1}{2}(z_{31,3} + z_{12} + z_{23,23} + z_{32,11} + z_{13,1123} + z_{21,113}) + \frac{1}{2}(z_{21,3} + z_{32} + z_{13,23} + z_{12,11} + z_{23,1123} + z_{31,113})$. The gravitational wave contribution to U is derivable from the relation $U = \gamma_{,1} - \beta$. The noise power spectrum for U is $S_U = [16\sin^2(\pi f L) + 8\sin^2(2\pi f L) + 16\sin^2(3\pi f L)]S_y^{\textit{proof mass}} + [4\sin^2(\pi f L) + 8\sin^2(2\pi f L) + 4\sin^2(3\pi f L)]S_y^{\textit{optical path}}$.

The $(\alpha, \beta, \gamma, \zeta)$ combinations are the simplest independent linear combinations of the Doppler data which do not contain laser or optical bench noises. These each involve all the y_{ij} and are 'six-pulse' gravity wave combinations; α (β, γ being given by cyclical index permutation) is $\alpha = y_{21} - y_{31} + y_{13,2} - y_{12,3} + y_{32,12} - y_{23,13} - \frac{1}{2}(z_{13,2} + z_{13,13} + z_{21} + z_{21,123} + z_{32,3} + z_{32,12}) + \frac{1}{2}(z_{23,2} + z_{23,13} + z_{31} + z_{31,123} + z_{12,3} + z_{12,12})$. The noise power spectrum of α is $S_\alpha = [8\sin^2(3\pi f L) + 16\sin^2(\pi f L)]S_y^{\textit{proof mass}} + 6S_y^{\textit{optical path}}$.

The symmetric data combination, ζ, having the property that each of the y_{ij} enters exactly once and is lagged by exactly one of the one-way light times, is $\zeta = y_{32,2} - y_{23,3} + y_{13,3} - y_{31,1} + y_{21,1} - y_{12,2} + \frac{1}{2}(-z_{13,21} + z_{23,12} - z_{21,23} + z_{31,23} - z_{32,13} + z_{12,13}) + \frac{1}{2}(-z_{32,2} + z_{12,2} - z_{13,3} + z_{23,3} - z_{21,1} + z_{31,1})$. ζ also has a six-pulse response to gravitational radiation. The noise power spectrum is $S_\zeta = 24\sin^2(\pi f L)\, S_y^{\textit{proof mass}} + 6S_y^{\textit{optical path}}$.

4. Gravitational wave sensitivities

The above information can be used to compute sensitivity for a general noise-cancelling configuration. Assume that LISA is in the ($\hat{u}$, $\hat{v}$) plane and the GW has wavevector $\hat{k}$. The angular coordinates of the source are (R, D), where R is the angle between $\hat{u}$ and the projection of $\hat{k}$ onto the LISA plane. Similarly, D is the angle between $\hat{k}$ and the ($\hat{u}$, $\hat{v}$) plane. To evaluate the dot products involved in the GW response in terms of the angular coordinates of the source, we adopt an orthonormal frame attached to the wave [7]: $\hat{k}$ is the direction of wave propagation, $\hat{i}$ is along positive-going D and $\hat{j}$ is along the positive-going R direction. Thus $\hat{i} = -\cos(R)\sin(D)\hat{u} - \sin(R)\sin(D)\hat{v} + \cos(D)\hat{w}$; $\hat{j} = -\sin(R)\hat{u} + \cos(R)\hat{v}$; and $\hat{k} = -\cos(R)\cos(D)\hat{u} - \sin(R)\cos(D)\hat{v} - \sin(D)\hat{w}$.

To calculate the Ψ_is, we need $\hat{k} \cdot \hat{n}_i$. By convention on the orientation of the $\hat{n}_i$, $l\hat{p}_3 + L_1\hat{n}_1 - l\hat{p}_2 = 0$, implying $\hat{n}_1 = l(\hat{p}_2 - \hat{p}_3)/L_1$ and so on for $\hat{n}_2$ and $\hat{n}_3$ by index permutation. The GW metric perturbation produces Doppler events given by equation (3). Let a_1 be the angle between $\hat{n}_1$ and $\hat{u}$, so that $\hat{n}_1 = \cos(a_1)\hat{u} + \sin(a_1)\hat{v}$. Since the source direction is $-\hat{k}$, n_1 dotted into $(\hat{i}, \hat{j}, \hat{k})$ produces: $\hat{n}_1 \cdot \hat{i} = -\sin(D)\cos(a_1 - R)$; $\hat{n}_1 \cdot \hat{j} = \sin(a_1 - R)$; $\hat{n}_1 \cdot \hat{k} = -\cos(D)\cos(a_1 - R)$. Thus the dot products required for the computation of Ψ_1 are $\hat{n}_1 \cdot \boldsymbol{e}_+ \cdot \hat{n}_1 = \sin^2(D)\cos^2(a_1 - R) - \sin^2(a_1 - R)$ and $\hat{n}_1 \cdot \boldsymbol{e}_x \cdot \hat{n}_1 = -\sin(a_1 - R)\cos(a_1 - R)\sin(D)$. The other Ψ_i are obtained, as usual, via cyclic index permutation.

The Ψ_i are now defined in terms of the orientation of a spacecraft link ($\hat{n}_i$) and the source angular coordinates (R, D). Suppose a monochromatic gravitational wave is incident with general elliptical polarization: $h_+ = H\sin\Gamma\sin(\omega t + \phi)$ and $h_\times = H\cos\Gamma\sin\omega t$, where H characterizes the strength of the wave and (Γ, ϕ) define its polarization state. Γ and ϕ are related to coordinates on a Poincaré sphere [8]. Then $\Psi_i(t) = \left[H/[2(1 - (\hat{k}\cdot\hat{n}_i)^2)]\right][P_i\cos(\omega t) + Q_i\sin(\omega t)]$, where $P_i = \sin\Gamma\sin\phi(\sin^2 D\cos^2(a_i - R) - \sin^2(a_i - R))$ and $Q_i = (\sin\Gamma\cos\phi)(\sin^2 D\cos^2(a_i - R) - \sin^2(a_i - R)) - 2\cos\Gamma\sin(a_i - R)\cos(a_i - R)\sin D$. These expressions for Ψ_i can be substituted into equations (1) and (2) to obtain expressions for y_{ij}^{gw}. For a sinusoidal GW excitation, the linear responses y_{ij}^{gw} are also sinusoidal. The y_{ij}^{gw} are then used to produce expressions for GW responses of the laser-noise-cancelling combinations; for example, substitute into equation (4) to obtain X^{gw}.

To compute the RMS gravitational wave response of each α, ζ, X, etc combination, we use a Monte Carlo technique [9]. At each Fourier frequency in the $\sim 10^{-4}$ Hz to $\sim 10^{-1}$ Hz LISA band, we average over source directions (uniformly distributed on the celestial sphere) and polarization states (uniformly distributed on the Poincaré sphere). The averaging is done with 2500 (source position, polarization state) pairs per Fourier frequency bin. The nominal LISA configuration is an almost equilateral triangle; we took $L_i = 16.67$ light seconds for the calculations here. The one-sided spectra of proof mass and optical path noise on a single link are currently expected to be 3×10^{-15} m s^{-2} Hz$^{-1/2}$ and 20×10^{-12} m Hz$^{-1/2}$, respectively [10]. Converted to Doppler spectra on a one-way link, these correspond to $S_y^{proof\ mass} = 2.5 \times 10^{-48}[f/1\ \mathrm{Hz}]^{-2}$ Hz^{-1} and $S_y^{optical\ path} = 1.8 \times 10^{-37}(f/1\ \mathrm{Hz})^2$ Hz^{-1}. In the sensitivity calculations here, we assume that the optical path noise has the same transfer function as the shot noise.

LISA sensitivity is conventionally taken to be the strength of a sinusoidal GW required to give SNR = 5 in a 1 yr integration time. Thus we computed $5\sqrt{S_j(f)B}$/(RMS gravitational wave response for data combination j), where j is α, ζ, X, etc and $B = 1$ cycle/year. Figure 1 shows the sensitivity for the eight-pulse combinations X, E, P and U. (In this equilateral case and for sensitivity averaged over the sky and polarization states, the

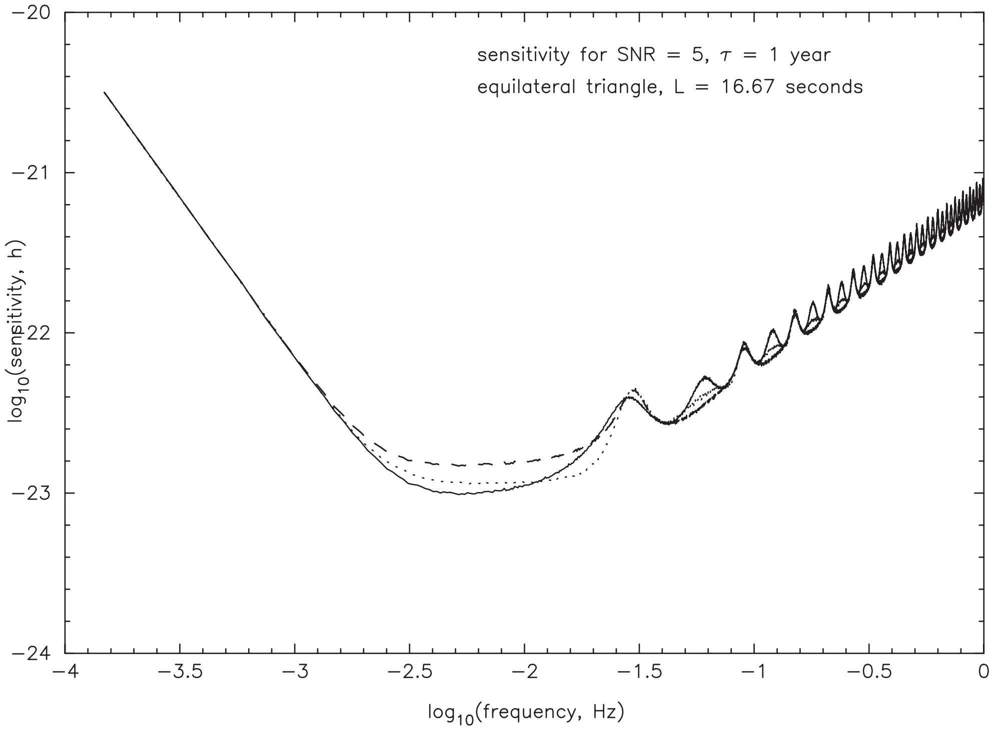

Figure 1. Sensitivity (SNR $= 5$, $\tau = 1$ yr) of Michelson (X, full), beacon (P, broken), monitor (E, broken) and relay (U, dotted) data combinations. These all cancel laser phase noise and optical bench motion noise and have eight-pulse GW responses. GW signals have been averaged over the sky and over polarization states; one-sided spectra of single-proof-mass noise and single-link optical path noise used were 3×10^{-15} m s^{-2} Hz$^{-1/2}$ and 20×10^{-12} m Hz$^{-1/2}$ (see text). Equal arm lengths (5 000 000 km) are assumed; in the equal-arm configuration and for this figure-of-merit, E and P have essentially the same sensitivity.

sensitivities of P and E are essentially equal.) The sensitivity curve for X can be compared with calculations where LISA is modelled as a rigid, equal-arm one-bounce interferometer. (These two sensitivities—for X and 'S', a one-bounce, equal-arm Michelson—should be the same.) The agreement [11, 12] is excellent when the same noise spectra are used. Figure 2 shows the sensitivity calculation for the six-pulse combinations α and ζ, under the same assumptions.

These sensitivities were computed based on shot and proof-mass noise only. Inclusion of expected confusion noise due to galactic binaries (e.g. [10], figure 1.3) would affect the low-frequency band edges, with small effects on the 3 dB bandwidths and the best sensitivities of each combination.

5. Concluding comments

Laser phase noise and optical-bench-motion noise-cancelling combinations can be built up from time-shifted linear combinations of one-way Doppler links tracking proof-mass pairs. The GW signal and residual instrumental noise contributions to these combinations can be analysed using signal and noise transfer functions of individual links to obtain the aggregate signal and noise of the laser and optical-bench cancelling combinations.

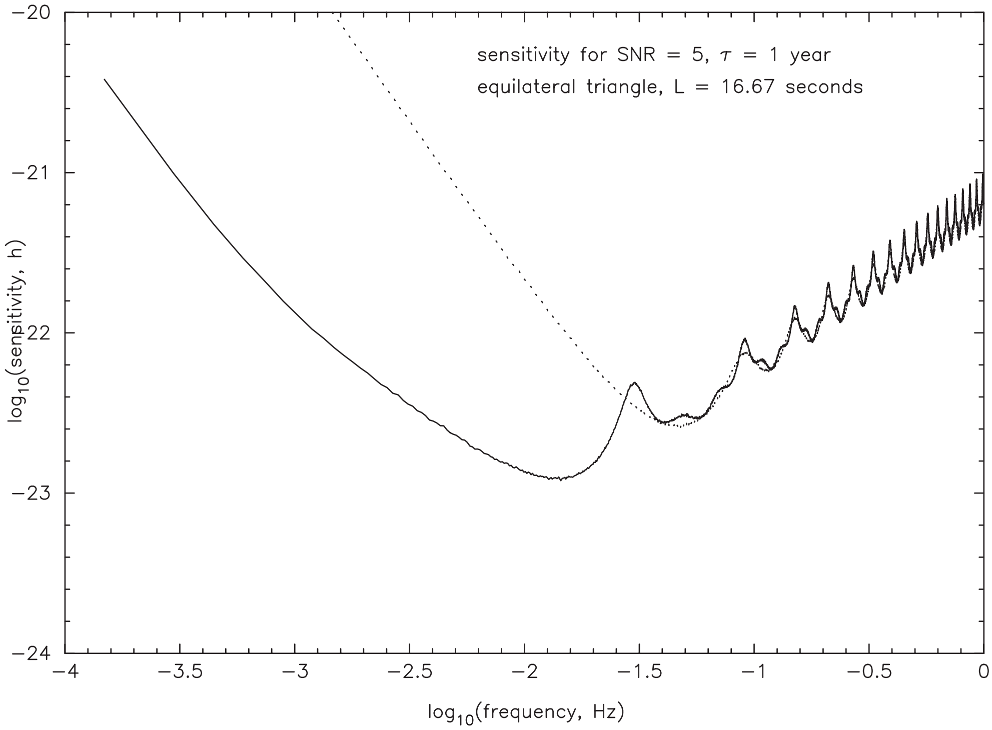

Figure 2. As in figure 1, but for the six-pulse noise-cancelling combinations α (full curve) and ζ (dotted curve).

We presented LISA sensitivity calculations for a few of these noise-cancelling configurations. More complicated noise-cancelling combinations can be built up from linear combinations of, for example, $X, Y, Z, \alpha, \beta, \gamma$, etc; these more complicated combinations may have utility in tailoring the response to specific sources or to compensate for possibly unequal noises in the as-flown LISA hardware [2]. The principal assumptions in this calculation were that LISA was a rigid equal-arm (5 000 000 km) system and that all six lasers have the same centre frequency. (If these are not strictly true, the laser-noise-elimination combinations will not completely eliminate optical-bench-motion noise.)

Finally, this analysis was only for sinusoidal waves and considered only sensitivity averaged over the sky and over elliptical polarization states as the figure-of-merit. Truly optimum combinations will probably depend on instrumental considerations (e.g. differing qualities of data streams with the as-flown hardware) and on the frequency, polarization state and source direction of the GWs actually detected.

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