

Average phase calculations with near-field diffraction algorithms for the Space Interferometry Mission

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ABSTRACT

This work reports on the computation of the average phase of a beam over an optical element via discrete Fourier transform techniques. The objective is to develop accurate diffraction models for the Space Interferometry Mission (SIM). Applications related to SIM include calibration of metrology measurements, evaluation of cornercube diffraction effects, and others. The algorithms that are used to compute the field are described and numerical tests that assess their accuracy are presented.

Keywords: near-field diffraction, discrete Fourier Transform, Goertzel-Reincsh algorithm

1. INTRODUCTION

With the operating parameters of SIM, however, deviation from geometrical optics is significant. It is expected that a lot of components will operate in the near-field regime. Ray-tracing is not adequate anymore and, therefore, the development of near-field diffraction models becomes necessary. Additionally, the goal of high precision astrometry requires extremely high level of accuracy of the optical models at a wide range of Fresnel numbers.

The present work reports on the accuracy of near-field diffraction algorithms that are based on discrete Fourier transform techniques. This work is part of the modeling efforts for the Space Interferometry Mission (SIM), a part of NASA's Origins program. Specific applications that require near-field diffraction modeling are calibration of measurements of the internal metrology system, evaluation of diffraction effects in cornercubes, starlight system modeling, *etc.* The accuracy requirements are particularly high and must be satisfied over a wide range of propagation distances. The error in the computation of the average phase, as a fraction of the wavelength, cannot exceed 20 picometers.

The quantity of interest is the average phase of the field over an optical element, usually a detector. In the sequel, calculations for the average phase are presented with algorithms that employ discrete Fourier transform techniques for the reconstruction of the optical field. Such algorithms are very popular because they combine accuracy, robustness, and speed. They are also easy to implement. The numerical tests are performed on the simplest possible geometry so that comparisons with closed-form solutions for the field can be made. It turns out that the desired level of accuracy can be achieved with proper choice of algorithm, over the entire range of the parameter of interest.

2. REPRESENTATIONS OF THE OPTICAL FIELD

It is assumed that an aperture is located at the plane $z = 0$, emitting plane waves of wavelength λ . The optical field, U , is assumed to be governed by the paraxial equation,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2 \right) U(x, y, z) = 0, \quad (1)$$

where $k = 2\pi/\lambda$ is the wavenumber. At the boundary $z = 0$, the field is equal to a given wavefront $U(\xi, \eta)$ inside the aperture, and it is identically zero outside.

One way to solve this equation is the angular spectrum method, which employs standard Fourier transform techniques, Goodman [1]. This method yields,

$$U(x, y, z) = e^{jkz} \iint_{-\infty}^{+\infty} \mathcal{F}(f_x, f_y, 0) e^{-j\pi\lambda z(f_x^2 + f_y^2)} e^{j2\pi(f_x x + f_y y)} df_x df_y, \quad (2)$$

where $\mathcal{F}(f_x, f_y, 0)$ is the Fourier transform of the field at the boundary $z = 0$. In the literature it is referred to as the angular spectrum of the field.

Another approach to solve this equation is via Green's function method. It is also called the "point source" method. The result is, [1],

$$U(x, y, z) = \frac{e^{jkz}}{j\lambda z} e^{\frac{jk}{2z}(x^2 + y^2)} \iint_{-\infty}^{+\infty} U(x, y, 0) e^{\frac{jk}{2z}(\xi^2 + \eta^2)} e^{\frac{j2\pi}{\lambda z}(x\xi + y\eta)} d\xi d\eta. \quad (3)$$

The two equations are equivalent, *i.e.*, they result in identical fields.

3. CALCULATION OF THE AVERAGE PHASE

The quantity of interest is defined as the phase of the integral of the field over Δ ,

$$\varphi \equiv \tan^{-1} \left(\frac{\text{Im}(I)}{\text{Re}(I)} \right) \quad (4)$$

$$I \equiv \int_{\Delta} U(x, y, z) dx dy. \quad (5)$$

The calculation of the average phase requires computation of the field and, subsequently, integration of the computed field.

The evaluation of the diffraction integral (2) can be performed with direct application of FFT. This approach results in a powerful and robust solver. The computational cost of the method is proportional to the cost of two 2-D FFT's. Semi-analytical guard-band requirements for the elimination of aliasing and energy spill-over into the computation domain have been introduced by Sziclas & Siegman, [2], Mansuripur, [3], and others.

In principle, FFT may also be employed for equation (3). This approach is even faster because it requires only one 2-D Fourier transform. However, the scaling factor $1/\lambda z$ that appears in the Fourier kernel introduces two significant computational problems. The first one is that the scaling factor does not allow evaluation of the field at the same (x, y) pairs that the field is known at the aperture. Numerical interpolation is, therefore, necessary if one is seeking the values at the same (x, y) pairs.

The second problem is associated with the resolution at the detector plane. Assume that the grid at the aperture consists of $N \times N$ points. Let $\Delta\xi$ be the sampling interval at the aperture and Δx the sampling interval at the detector plane. The relationship between the two intervals is $\Delta x = \lambda z / (N\Delta\xi)$. In other words, Δx is inversely proportional to $\Delta\xi$. The two sampling intervals can be simultaneously reduced only if N is increased. It turns out that to achieve the desired degree of accuracy, N has to be ampler than current computational resources can handle.

A more efficient approach for the numerical integration of (3) is to employ recurrence formulae that hold for trigonometric functions. The first step is to discretize the integral and approximate it as a Fourier series. The summation of the series can be achieved via the Goertzel-Reinsch algorithm, Stoer & Bulirsch, [4]. This algorithm performs evaluation of trigonometric series of the type $\sum f_m e^{jm\theta}$. It is based on the fact that for the sums defined by

$$C_\rho = \frac{1}{\sin\theta} \sum_{m=j}^{N-1} f_m \sin(m - \rho + 1)\theta, \quad \rho = 0, 1, \dots, N - 1. \quad (6)$$

the following recurrence relation holds,

$$C_\rho = f_\rho + 2C_{\rho+1} \cos\theta - 2C_{\rho+2}, \quad \rho = N - 1, N - 2, \dots, 0. \quad (7)$$

This relation has to be employed in descending order with initial values $C_N = 0$, $C_{N+1} = 0$. The series of interest are then given by

$$\sum_{m=0}^{N-1} f_m e^{jm\theta} = f_0 + C_1 \cos\theta - C_2 + jC_1 \sin\theta. \quad (8)$$

In the sequel, this method for computing (3) will be referred to as the *direct method*.

The critical parameter of the problem is the Fresnel number, defined as

$$Fr = \frac{w^2}{\lambda z}, \quad (9)$$

where w is the characteristic semi-length of the aperture. It is easy to verify that the integrand in (2) becomes more oscillatory as the Fresnel number decreases, while the opposite is true for the integrand in (3). Hence it is expected that with fixed resolution, the angular spectrum method will be more accurate for high Fresnel numbers, while the direct method will be more accurate for small Fresnel numbers.

Once the field has been computed, the average phase can be obtained by numerical integration of the field over the detector Δ . This integration can be performed by standard two-dimensional quadrature algorithms, such as Simpson's rule. Other choices are, of course, available. Probably the most accurate method is Gaussian quadrature but it requires knowledge of the field in non-uniform sampling. Therefore, it cannot be employed on fields computed by FFT-based algorithms. The results presented below have been obtained by Simpson's rule.

4. NUMERICAL TESTS

The accuracy of the algorithms described above was explored through a series of numerical tests. Results for some of these tests are presented in this section. The tests consisted of beam propagation from a square aperture to a square detector. For this particular problem the field is known analytically, [1]. A very accurate estimation for the average phase can be obtained by Gaussian quadrature of the analytic solution of the field. Numerical convergence tests showed that accuracy up to the 7th significant digit can be achieved on a grid that consists of 512 points on each direction. Results obtained in this fashion are used for testing the accuracy of the two algorithms described above. The error in the calculation of the average phase is measured as a portion of the wavelength. For example, an error equal to π is said to be an error of half a wavelength. The error is measured in picometers. The grid size for the angular spectrum method was 2048×2048 points with the guard-band ratio equal to 4 at the aperture. The grid size for the direct method was 512×512 points at both aperture and detector.

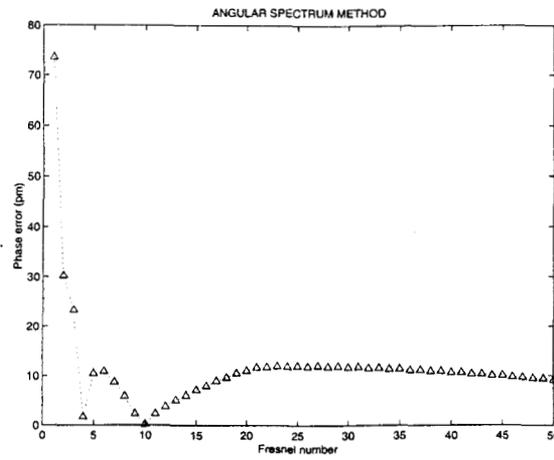


Figure 1. Average phase error of the angular spectrum method.

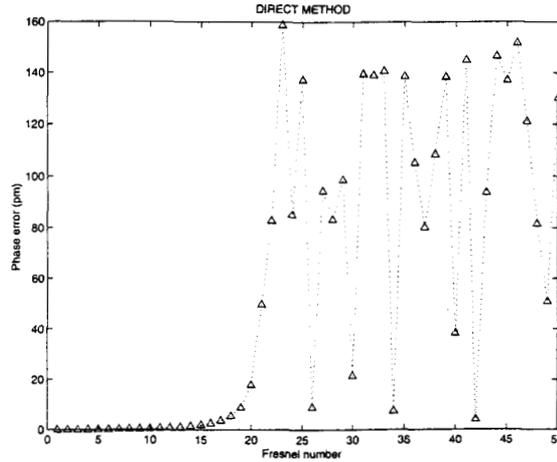


Figure 2. Average phase error of the direct method.

For the first test the width of the aperture was $2w = 3 \text{ cm}$, while the width of the detector was $2W = 6 \text{ cm}$. The wavelength was $\lambda = 0.7 \mu\text{m}$. Results on the errors of the two methods as a function of the Fresnel number are shown in figures 1 and 2. The error of the angular spectrum method was smaller than the error tolerance of 20 picometers for $Fr = 4$ and higher. In contrast the direct method gave acceptable results for $Fr = 20$ and lower. Therefore, there is an overlapping region where both algorithms yield satisfactory results. Further, for Fresnel numbers larger than approximately 19, the angular spectrum method is more accurate than the direct method.

On a different test, the aperture width was set at $2w = 5 \text{ mm}$ and the detector width at $2W = 10 \text{ mm}$. The wavelength was $\lambda = 1.3 \mu\text{m}$. The propagation distance varied between $z = 10 \text{ m}$ and 12 m . The Fresnel number at $z = 10 \text{ m}$ is about $Fr = 0.48$. The errors of the two methods as a function of the propagation distance are plotted in figure 3 and 4.

Results obtained with four different resolutions with the angular spectrum method, are shown in figure 3. The grid sizes were 2048, 4096, 8192, and 32768 points per direction, respectively. These results were obtained at Caltech's HP/Exemplar SPP200, a 256-processor machine with 64 Gigabytes of memory. The code was parallelized so that the data size could be distributed over 128 processors. The guard-band ratio

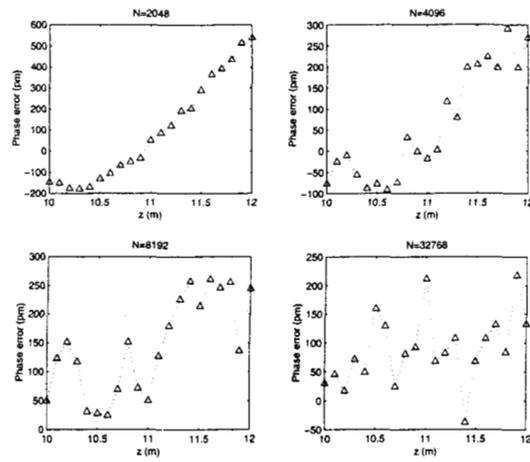


Figure 3. Average phase error of the angular spectrum method.

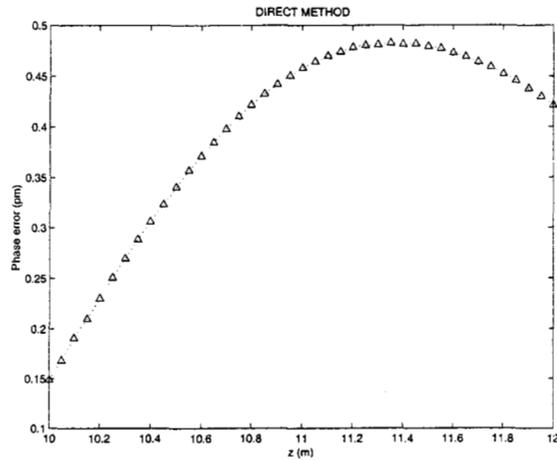


Figure 4. Average phase error of the direct method.

was set at 16, in order to minimize aliasing. The performance of the method in this test case, however, is not satisfactory. The errors are too high for the purposes of SIM, even at the finest resolution. On the other hand, the results obtained by the direct method for this test case are extremely accurate, figure 4. The error has been kept below 1 picometer. In particular, the error peak was 0.49 picometers at $z = 11.35 \text{ m}$.

5. CONCLUSIONS

The present study was focused in the accuracy of Fourier-based algorithms for the evaluation of the average phase of a beam in the near field. The error tolerance in the computations is 20 picometers. Numerical tests showed that the angular spectrum approach with FFT is a reliable method for high Fresnel numbers, generally higher than 15. The method based on direct integration of the diffraction integral (3) via the Goertzel-Reinsch algorithm yields satisfactory results for Fresnel numbers smaller than 20. The error in the computation of the average phase can always be kept below the tolerance level with proper choice of one of these two algorithms.

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