# OPTIMAL CONTROL STRATEGIES FOR IMAGING USING FORMATION FLYING SPACECRAFT

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Abstract: We delineate on the optimal control strategies for the *uv*-plane coverage and the associated combinatorial optimization problems. The results are represented in the context of the separated spacecraft optical interferometry ST3 mission.

Keywords: optical interferometry, *uv*-plane coverage, optimal control, combinatorial optimization

## 1. INTRODUCTION

Space-borne optical interferometry holds the promise of revolutionizing our understanding of the origin and evolution of the planetary systems. By synthesizing the image of a star or possibly its orbiting planets through interfering light gathered by two or more collectors, a unique window into characterizing size, temperature, and orbital parameters is provided (Davis, 1985).

Imaging using optical interferometers is through a process called *uv*-plane coverage (see §2.1). The paper establishes a connection between optimal control strategies for the *uv*-plane coverage on one hand, and the mission fuel and time allocation and the underlying combinatorial optimization problem on the other. To make the presentation specific, we focus on a particular planned NASA optical interferometry mission called the Space Technology 3 (ST3). By doing so we find an opportunity to describe in some detail the extra transformation needed to translate *uv*-plane coverage to the actual spacecraft movements in the physical space. In the ST3 case, these movements constitute a paraboloid.

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The paper addresses the following questions for ST3:

(1) Given a time and fuel allocation, how many stars can be imaged (assuming that the coverage proceeds from one star to the next, and that the specified *uv* points for a given star have already been sequenced)?
(2) Given $N$ stars and a particular set of *uv* points for each, what is the optimal fuel strategy for covering them (given a certain allowable mission life time)?

The first question is addressed by optimal control laws which are parameterized in terms of the mission life time and fuel allocation. The second question leads us directly to a combinatorial optimization problem.

A few words on the notation. The mass of the spacecraft is denoted by $M$ (kilograms), its available fuel by $F$ (kilograms), and the Isp of its thrusters by $I$ (sec). The thrusters are assumed to have a maximum thrust level of $T$ (Newton); $\mathbf{a}$ denotes a vector of length $a$- when $\mathbf{a}$ is represented in the frame $\mathcal{F}$, we write $(\mathbf{a})_{\mathcal{F}}$. $g$ denotes the earth's gravitation constant.

We assume that the spacecraft is in free space, its mass is fixed during the particular interval of interest (however it can be regularly updated),

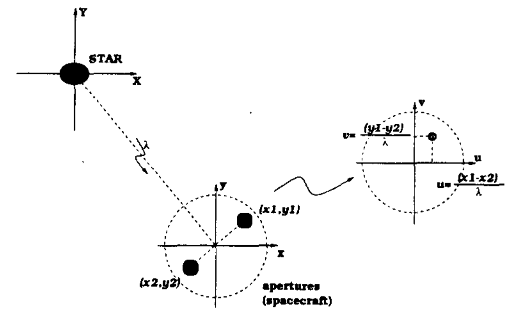


Fig. 1. Imaging a star through optical interferometry

and that the disturbance forces are negligible. We also assume that each spacecraft is equipped with on/off thrusters.

## 2. PRELIMINARIES

### 2.1 *What is the uv-plane coverage?*

Imaging a star can be thought as the evaluation of its intensity function in the frame attached to the star. The van Cittert-Zernike formula

$$I(X,Y) = \int_u \int_v \mu(u,v) e^{i2\pi(uX+vY)} dv\, du,$$

relates the evaluation of $I$ to the inverse Fourier transform of a function which can be measured by optical instruments. This function, represented by $\mu$, is called the mutual coherence function (Born and Wolf, 1997). Here one has,

$$u = \frac{1}{\lambda}(x_1 - x_2) \quad \text{and} \quad v = \frac{1}{\lambda}(y_1 - y_2), \quad (2.1)$$

where $\lambda$ is the wavelength of the light emitted by the star- $(x_1, y_1)$, $(x_2, y_2)$ are the positions of the aperture in the physical space. See Figure 1.

Imaging a star thus amounts to the evaluation of the mutual coherence function over a disk of infinite radius in the $uv$-plane. By choosing a disk of finite radius and sampling (evaluating the mutual coherence function) at a finite number of $uv$-points, $I$ is approximated by keeping a range of its frequency content.

Suppose that we place each of the aperture on a separate spacecraft. The process of moving the two spacecraft (aperture) in a formation (Mesbahi and Hadaegh, 1999), (Wang and Hadaegh, 1996), in order to sample a specified points in the $uv$-plane is called the $uv$-plane coverage.

Selecting the $uv$-points based on some apriori assumptions about the source (star) is an interesting problem by its own right. For the purpose of the present discussion, we assume that for each star, the $uv$-points that need to be sampled have

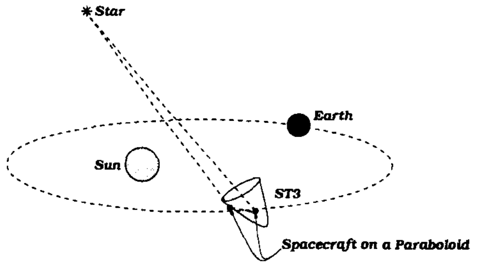


Fig. 2. ST3 in a helio-centric orbit

already been specified: for $N$ stars, this amounts to a list of the form,

$$\underbrace{(u_1, v_1), \ldots, (u_{n_1}, v_{n_1})}_{\text{star 1}}, \ldots$$
$$\underbrace{(u_1, v_1), \ldots, (u_{n_2}, v_{n_2})}_{\text{star 2}}, \ldots$$
$$\underbrace{(u_1, v_1), \ldots, (u_{n_N}, v_{n_N})}_{\text{star } N},$$

where $n_j$ is the number of $uv$ points for star $j$.

### 2.2 *The Geometry of the ST3 Mission*

The geometry of every optical interferometry mission is dictated mainly by the path lengths that the starlight travels to where the image is finally synthesized. We briefly discuss the configuration of ST3 and and the rationale for it proposed parabolic geometry.

Recall that a parabola is a set of points that are equally distant ($p$) from a fixed point (focus) and a fixed line (directrix). The axis of the parabola is perpendicular to the directrix and passes through the focus. The point on the parabola axis, half way between the focus and and the directrix is called the vertex. If the $z$-axis is the axis of the parabola and the $x$-axis is perpendicular to it, intersecting at the vertex, then the equation of the parabola is $z = \frac{1}{2p}x^2 = \frac{1}{4f}x^2$. The paraboloid of rotation is formed by rotating the parabola about its axis i.e., $z = \frac{1}{4f}(x^2 + y^2)$. The orbit chosen for the ST3 mission is earth-trailing (Figure 2). One of the spacecraft, the combiner, carries the instruments necessary for collecting and combining light beams. The other spacecraft, the collector, carries the instruments for collecting light only. The light gathered at the collector is redirected to the combiner and interfered with the light gathered at the combiner. Interferometry requires that the distance that the two light beams travel are equal. We call this requirement the zero-delay condition. By placing the combiner at the focus of paraboloid and moving the collector on its perimeter, it can be shown that the zero-delay condition is met by providing an internal delay of length $2f$ in the combiner (Figure 3).

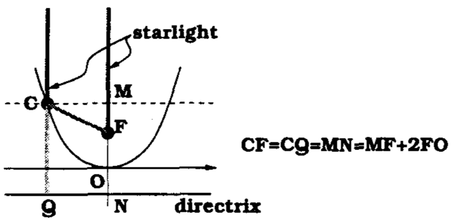


Fig. 3. Why a paraboloid for ST3?

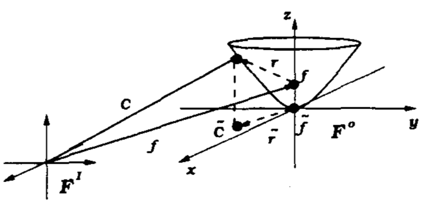


Fig. 4. Projections from the ST3 paraboloid to the $uv$-plane

### 2.3 *Mapping the uv-plane to the ST3 Paraboloid*

We are given a set of $uv$ points which need to be sampled for a given star. Since the spacecraft are required to move on a paraboloid, we would like to derive an explicit expression which relates the movements between the $uv$ points to the corresponding spacecraft movements in the physical space.

Let $\mathcal{F}^I$ denote the inertial reference frame. Let $\mathcal{F}^O$ denote the reference frame attached to the ST3 paraboloid, with its $z$-axis parallel to the axis of observation. The $xy$-plane of $\mathcal{F}^O$ shall be denoted by $\mathcal{F}^O_{xy}$. Let $M^O_I$ denote the direction cosine matrix that associates vectors represented in $\mathcal{F}^I$ to those that are represented in $\mathcal{F}^O$; similarly $(M^O_I)_{xy}$ denotes the direction cosine matrix associated with the respective $xy$-planes.

Denote by $\mathbf{f}$ and $\mathbf{c}$, the locations of the combiner and the collector in the inertial frame. Define,

$$\mathbf{r} := \mathbf{c} - \mathbf{f}.$$

Let us denote the projection of $\mathbf{r}$ onto the $xy$-plane of $\mathcal{F}^0$ by $\tilde{\mathbf{r}}$.

Then using (2.1) one has,

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} - \lambda \begin{bmatrix} u \\ v \end{bmatrix}.$$

Motivated by the required geometry of the ST3 and the associated $uv$-plane, we proceed to let,

$$(\tilde{\mathbf{r}})_{\mathcal{F}^O_{xy}} = \lambda \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \tilde{r}_1 \\ \tilde{r}_2 \end{bmatrix}, \ (\tilde{\mathbf{f}})_{\mathcal{F}^O_{xy}} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix},$$

$$\text{and} \quad (\tilde{\mathbf{c}})_{\mathcal{F}^O_{xy}} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix};$$

refer to Figure 4. Then,

$$(\tilde{\mathbf{r}})_{\mathcal{F}^O} = \begin{bmatrix} \tilde{r}_1 \\ \tilde{r}_2 \\ 0 \end{bmatrix}, \quad (\mathbf{f})_{\mathcal{F}^O} = \begin{bmatrix} 0 \\ 0 \\ f \end{bmatrix},$$

$$\text{and} \quad (\mathbf{c})_{\mathcal{F}^O} = \begin{bmatrix} \tilde{r}_1 \\ \tilde{r}_2 \\ \frac{1}{4f}(\tilde{r}_1^2 + \tilde{r}_2^2) \end{bmatrix}.$$

The relative position of the collector with respect to the combiner is therefore,

$$\begin{aligned} (\mathbf{r})_{\mathcal{F}^I} &= (M^O_I)^T(\mathbf{c} - \mathbf{f})_{\mathcal{F}^O} \\ &= (M^O_I)^T \begin{bmatrix} \tilde{r}_1 \\ \tilde{r}_2 \\ \frac{1}{4}(\tilde{r}_1^2 + \tilde{r}_2^2) - f \end{bmatrix}. \end{aligned}$$

In order to simplify the presentation we assume that $M^O_I$ is identity and that $\lambda = 1$. Thus the vector $\mathbf{r}$ is simply,

$$(\mathbf{r})_{\mathcal{F}^I} = \begin{bmatrix} u \\ v \\ \frac{1}{4f}(u^2 + v^2) - f \end{bmatrix}.$$

Specifying a set of $(u, v)$ points for imaging a given star therefore directly translates to a sequence of $\mathbf{r}$'s in the Euclidean space. In other words, $uv$-plane coverage for $N$ stars is simply a collection,

$$\underbrace{\mathbf{r}_1, \ldots, \mathbf{r}_{n_1}}_{\text{star 1}}, \ldots \underbrace{\mathbf{r}_1, \ldots, \mathbf{r}_{n_2}}_{\text{star 2}}, \ldots \underbrace{\mathbf{r}_1, \ldots, \mathbf{r}_{n_N}}_{\text{star } N},$$

where $n_j$ is the number of $uv$ points scheduled to be sampled for star $j$.

## 3. HOW MANY SOURCES CAN ST3 OBSERVE?

In this section we address the following problem: Given a time and fuel allocation, how many sources can ST3 observe, when one or two spacecraft moves are allowed? To address this question, we make two simplifying assumptions, which in a sense, remove the underlying combinatorial character of the $uv$-plane coverage. We shall come back to the combinatorial nature of the problem in the next section. These assumptions are as follows:

1. The $uv$-plane coverage for $N$ stars proceeds from one star to the next.
2. For each star, the scheduled $(u, v)$ points have already been sequenced.

Under these two assumptions, we derive explicit formula for the number of sources which can be observed by ST3. Moreover, we specify the corresponding optimal control laws for each spacecraft movement and the underlying control architecture.

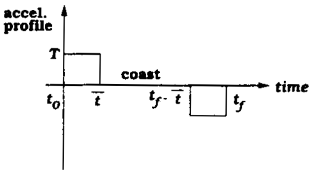


Fig. 5. The acceleration profile

### 3.1 *An optimal control strategy for moving one spacecraft*

Let the fuel depletion rate for each spacecraft at time $t$ be represented by,

$$\frac{dF(t)}{dt} = \begin{cases} -\gamma T & \text{when } F(t) > 0 \text{ and } \mathbf{u}(t) \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $\gamma = \frac{1}{Ig}$, $T$ is the maximum available thrust, $I$ is the thrusters' Isp, and $\mathbf{u}$ is the control force applied on the spacecraft.

We assume that the spacecraft has already been turned such that the thrust direction is parallel to the desired move, $\mathbf{d}$. We also consider the case where full thrust $T$ is applied on the spacecraft during the interval $[t_0, \bar{t}]$, coast during $[\bar{t}, t_f - \bar{t}]$, and then decelerate during $[t_f - \bar{t}, t_f]$, where $t_0$ and $t_f$ are the initiation and termination time of the movement (refer to Figure 5). This control law has the form of a bang-coast-bang control, a class which is known to contain an optimal fuel maneuver (Stengel, 1994).

Integrating the spacecraft equation of motion twice and plugging in the boundary conditions we obtain,

$$\bar{t}(d) = \frac{t_f(d)}{2} - \sqrt{\frac{t_f(d)^2}{4} - \frac{Md}{T}}, \qquad (3.2)$$

where $M$ denotes the mass of the spacecraft (either the collector or the combiner), and $d$ is the distance that the spacecraft has moved during the interval $[t_0, t_f]$.

The fuel usage for a movement of length $d$ is therefore,

$$F(d) = 2\gamma T \bar{t}(d)$$

and thus,

$$t_f(d) = \frac{F(d)}{2\gamma T} + \frac{2\gamma M d}{F(d)}. \qquad (3.3)$$

For a movement of length $d$, consider the following minimization problem,

$$J(F(d), t_f(d)) = F(d) + \lambda t_f(d) \rightarrow \min,$$

an optimal control problem, where the parameter $\lambda$ provides means of specifying the relative importance of fuel or time (we say more about the selection of $\lambda$ shortly). The optimal control problem is generically concerned about choosing a control law which is optimal for a given objective functional. Since the form of the control law (bang-coast-bang) has been specified, the optimization parameters are the coast interval and the maneuver completion time, determined by $\bar{t}(d)$ and $t_f(d)$.

Taking the first derivative of (3.4) with respect to $F(d)$ after replacing $t_f(d)$ with its equivalent expression (3.3), we obtain,

$$\begin{aligned} F^*(d) &= 2\gamma\sqrt{\frac{\lambda T}{2\gamma T + \lambda}}\sqrt{Md} \\ &= 2\gamma\sqrt{MTd}\beta(\lambda), \qquad (3.4) \end{aligned}$$

and

$$\begin{aligned} t_f^*(d) &= \frac{F^*(d)}{2\gamma T} + \frac{2\gamma M d}{F^*(d)} \qquad (3.5) \\ &= \sqrt{\frac{Md}{T}}(\beta(\lambda) + \frac{1}{\beta(\lambda)}), \qquad (3.6) \end{aligned}$$

where,

$$\beta(\lambda) = \sqrt{\frac{\lambda}{2\gamma T + \lambda}}; \qquad (3.7)$$

see Figure 6.

Recall that the $uv$-plane coverage translates directly to traversing a sequence of vectors in the Euclidean space. Suppose that the $(u, v)$ points for a given star have already been sequenced- that is, we are given the sequence $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$, which corresponds to the desired $uv$-plane coverage.

Let $d_{ij} = \|\mathbf{r}_i - \mathbf{r}_j\|$ $(i = 1, \ldots, n-1, j = i+1)$. Define,

$$D := \frac{1}{N}\sum_{j=1}^{N}\sum_{i=1}^{n_j}\sqrt{d_{ij}},$$

which is the average square root distance traveled by the formation per source.

Using (3.5)-(3.7) the total fuel and time required for the $uv$-plane coverage of $N$ stars using a spacecraft of mass $M$ is,

$$f_{\text{total}} = 2\gamma DN\sqrt{MT}\beta(\lambda),$$

and

$$t_{\text{total}} = DN\sqrt{\frac{M}{T}}(\beta(\lambda) + \frac{1}{\beta(\lambda)}).$$

Given that the maximum available thrust $T$, the Isp $I$, and the mass of the spacecraft $M$ are

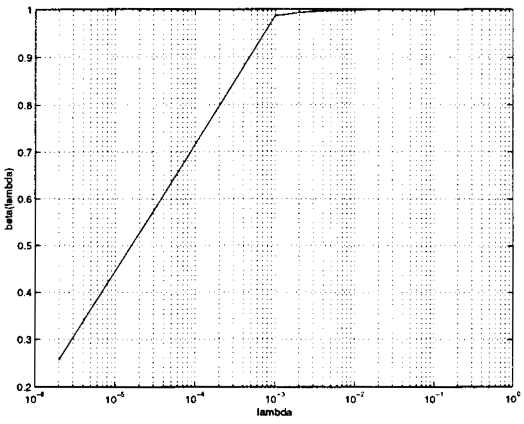


Fig. 6. $\beta(\lambda)$

fixed, $\lambda$ is the only parameter which can chosen to satisfy a given mission requirement. The choice of $\lambda$ effects:

- the acceleration/deceleration time, determined by $\bar{t}(d)$,
- the fuel consumption for a given desired repositioning of the spacecraft, and
- the time required to complete the maneuver for a given repositioning of the spacecraft.

How $\lambda$ effects these parameters is through the multiplier $\beta(\lambda)$. We note that when $\lambda \gg 2\gamma T$, $\beta(\lambda) \approx 1$.

Given $f_{\text{total}}$ and $t_{\text{total}}$, the number of sources which can be observed by ST3 is therefore,

$$N = \lfloor \min\{\frac{f_{\text{total}}}{2\gamma D\sqrt{mT}\beta(\lambda)}, \frac{t_{\text{total}}}{D\sqrt{\frac{m}{T}}(\beta(\lambda) + \frac{1}{\beta(\lambda)})}\}\rfloor.$$

If one would like the mission to run out of fuel and time at about the same time, let,

$$\frac{f_{\text{total}}}{t_{\text{total}}} = \frac{2\gamma\sqrt{T}\beta(\lambda)}{\frac{1}{\sqrt{T}}(\beta(\lambda) + \frac{1}{\beta(\lambda)})},$$

or, after some simplification,

$$\alpha := \frac{f_{\text{total}}}{t_{\text{total}}} = \frac{\gamma T\lambda}{\gamma T + \lambda};$$

this expression can be solved for $\lambda$; denote the solution by $\bar{\lambda}$.

More generally, given a fuel and time allocation, we can determine the number of sources which can be observed as follows:

(1) Let $\alpha = f_{\text{total}}/t_{\text{total}}$ and $\bar{\lambda} = (\alpha\gamma T)/(\gamma T - \alpha)$.
(2) Set $\beta(\bar{\lambda}) = \sqrt{\bar{\lambda}/(2\gamma T + \bar{\lambda})}$.
(3) The number of sources which can be observed is then,

$$N = \frac{f_{\text{total}}}{2\gamma D\sqrt{MT}\beta(\bar{\lambda})}.$$

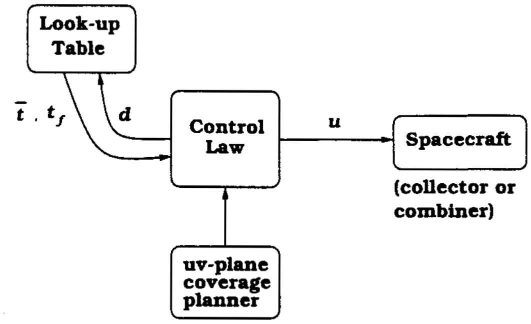


Fig. 7. $uv$-plane coverage with one spacecraft

The corresponding control law for a move of length $d$ can also be determined as follows:

(1) Find $t_f^*(d)$ using (3.7), i.e.,

$$t_f^*(d) = \sqrt{\frac{Md}{T}}(\beta(\bar{\lambda}) + \frac{1}{\beta(\bar{\lambda})}).$$

(2) The acceleration time parameter is then found using (3.9),

$$\bar{t}(d) = \frac{t_f^*(d)}{2} - \sqrt{\frac{(t_f^*(d))^2}{4} - \frac{Md}{T}},$$

The parameter $\bar{t}$ determines the form of the control law used for repositioning of the spacecraft for a move of length $d$. For example, given a predetermined set of moves, one would proceed to calculate the corresponding $\bar{t}$'s and have them organized in a lookup table as part of the control algorithm (see Figure 7).

### 3.2 *An optimal control strategy for moving both spacecraft*

In this section we consider the case where both the combiner and the collector are allowed to move for the $uv$-plane coverage. This case is particularly relevant, when due to some other mission considerations, it is desired that the area to mass ratio of both spacecraft remain close during the mission life time.

Let us denote the mass of collector and the combiner by $M_1$ and $M_2$, respectively; we let $M_1 = M$, and $M_2 = \theta M$, where $\theta > 1$.

Suppose that in order to move between two $uv$ points, the *formation* is required to travel a distance $d$. This means that the total movement of the collector and the combiner add up to $d$. Let the collector move $\delta d$ and the combiner move $(1-\delta)d$, where, $0 \leq \delta \leq 1$. According to (3.5)-(3.7) the corresponding fuel usage for the collector and the combiner would then be $2\gamma\sqrt{M\delta Td}\beta(\lambda_1)$ and $2\gamma\sqrt{M(1-\delta)Td}\beta(\lambda_2)$, respectively, where $\lambda_1$ and $\lambda_2$ are the optimization parameters in the optimal control laws for the collector and the combiner.

The total fuel usage for the $uv$-plane coverage of $N$ sources when moving both spacecraft is thus,

$$f_{\text{total}} = 2\gamma DN\sqrt{MT}(\sqrt{\delta}\beta(\lambda_1) + \sqrt{(1-\delta)\theta}\beta(\lambda_2)).$$

The total mission time would then be,

$$t_{\text{total}} = \max\{DN\sqrt{\frac{M\delta}{T}}(\beta(\lambda_1) + \frac{1}{\beta(\lambda_1)}), \\ DN\sqrt{\frac{M\theta(1-\delta)}{T}}(\beta(\lambda_2) + \frac{1}{\beta(\lambda_2)})\}.$$

If one requires that the collector and the combiner use the same amount of fuel during their respective moves and finish their maneuvers at the same time, we should let $\lambda_1 = \lambda_2$ and $\delta = \frac{\theta}{1+\theta}$.

Comparing the total fuel usage for the single spacecraft moves and the two spacecraft moves, we note that in the first case the fuel usage is determined by $\beta(\lambda)$ and in the second by,

$$(\sqrt{\delta} + \sqrt{(1-\delta)\theta}\beta(\lambda).$$

Of course the $\lambda$'s in the two cases (moving the collector only and moving both the combiner and the collector) do not have to be equal. If the two $\lambda$'s are *chosen* to be equal, since $(\sqrt{\delta} + \sqrt{(1-\delta)\theta} \geq 1$ when $\theta \geq 1$, moving both spacecraft always results in higher fuel usage (the mission life time shortens). However, it is best to choose $\bar{\bar{\lambda}} < \bar{\lambda}$ as we proceed to show ($\bar{\lambda}$ was defined in the previous section).

Setting $\theta(1-\delta) = \delta$, the time required for the $uv$-plane coverage when moving both spacecraft is,

$$t_{\text{total}} = DN\sqrt{(M\delta/T)}(\beta(\lambda) + \frac{1}{\beta(\lambda)}).$$

The corresponding fuel usage is then $f_{\text{total}} = 4DN\gamma\sqrt{MT\delta}\beta(\lambda)$. Similar to the single spacecraft case, if we require that the mission runs out of fuel and time at the same time, we end up solving for $\lambda$ in,

$$\alpha := \frac{f_{\text{total}}}{t_{\text{total}}} = \frac{2\gamma T\lambda}{\lambda T + \gamma}.$$

Denote the solution by $\bar{\bar{\lambda}}$, i.e., $\bar{\bar{\lambda}} = \frac{\alpha\gamma T}{2\gamma T - \alpha}$.

Note that $\bar{\bar{\lambda}} < \bar{\lambda}$ and that the ratio between the fuel usage for one spacecraft moves and the two spacecraft moves is,

$$\frac{\beta(\bar{\lambda})}{2\sqrt{\delta}\beta(\bar{\bar{\lambda}})}.$$

Thus the number of sources which the ST3 can observe with two spacecraft moves is,

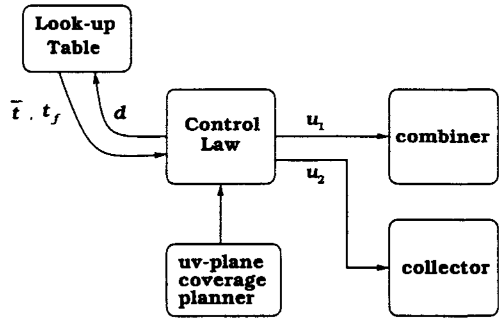


Fig. 8. $uv$-plane coverage with two spacecraft moves

$$N = \lfloor \frac{f_{\text{total}}}{4\gamma D\sqrt{MT\delta}\beta(\bar{\bar{\lambda}})} \rfloor.$$

The control architecture for the $uv$-plane coverage by moving both the collector and the combiner is shown in Figure 8.

## 4. CONCLUDING REMARKS

This paper considered the optimal $uv$-plane coverage for an optical interferometry mission. In this direction, we delineated on the relationship between the optimal control laws which are used during the $uv$-plane coverage, the allocated mission and fuel budget, and the number of stars which can be imaged. The results were presented in the context of the ST3 mission.

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