Quantum Metrology

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Heisenberg-limited measurement protocols can be used to gain an increase in measurement precision over classical protocols. Such measurements can be implemented using, e.g., optical Mach-Zehnder interferometers and Ramsey spectroscopes. We address the formal equivalence between the Mach-Zehnder interferometer, the Ramsey spectroscope, and a specific quantum logical gate. Based on this equivalence we introduce the quantum "Rosetta stone", and we describe a projective-measurement scheme for generating the desired correlations between the interferometric input states in order to achieve Heisenberg-limited sensitivity.

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I. INTRODUCTION

A generic classical interferometer has a shot-noise limited sensitivity that scales with $N^{-\frac{1}{2}}$. Here, N is either the number of particles passing through the interferometer during measurement time, or the number of times the experiment is performed with single-particle inputs [1,2]. However, when we carefully prepare quantum correlations between the particles, the inteferometer sensitivity can be improved by a factor of $\sqrt{N}$. That is, the sensitivity now scales with $1/N$. This limit is imposed by the Heisenberg uncertainty principle. For optical interferometers operating at several milliwatts, the quantum sensitivity improvement corresponds to an enhanced signal to noise ratio of eight orders of magnitude.

As early as 1981, Caves showed that feeding the secondary input port of an optical Mach-Zehnder interferometer with squeezed light yields a shot-noise lower than $N^{-\frac{1}{2}}$ (where N is now the average photon number) [3]. Also, Yurke *et al.* [4,5] showed in 1986 that a properly correlated Fock-state input for the Mach-Zehnder interferometer (here called the *Yurke state*) could lead to a phase sensitivity of $\Delta\phi \simeq O(1/N)$. This improvement occurred for special values of ϕ. Sanders and Milburn, and Ou generalized this method to obtain $1/N$ sensitivity for all values of ϕ [6,7].

Subsequently, Holland and Burnett proposed the use of *dual Fock states* (of the form $|N,N\rangle$) to gain Heisenberg limited sensitivity [8]. This dual-Fock-state approach opened new possibilities; in particular, Jacobson *et al.* [9], and Bouyer and Kasevich [10] showed that the dual Fock state can also yield Heisenberg-limited sensitivity in atom interferometry.

A similar improvement in measurement sensitivity can be achieved in the determination of frequency standards and spectroscopy; an atomic clock using Ramsey's separated-oscillatory-fields technique is formally equivalent to the optical Mach-Zhender interferometer. Here, the two $\pi/2$-pulses represent the beam splitters. Wineland and coworkers first showed that the best possible precision in frequency standard is obtained by using maximally entangled states [11]. Similarly, it was shown by one of us (JPD) that this improved sensitivity can be exploited in quantum gyroscopes [12].

Quantum lithography and microscopy is closely related to this enhanced sensitivity. In practice, the bottleneck for reading and writing with light is the resolution of the feature size, which is limited by the wavelength of the used light. In calssical optical lithography the minimum feature size is determined by the Rayleigh diffraction limit of $\lambda/4$, where λ is the wavelength of the light. It has been shown that this classical limit can be overcome by exploiting the quantum nature of entangled photons [13–18].

This principle works both ways: in classical optical microscopy too, the finest detail that can be resolved cannot be much smaller than the optical wavelength. using the same entangled-photons technique, it is possible to image the features substantially smaller than the wavelength of the light. The desired entangled quantum state for quantum interferometric lithography yielding a resolution of $\lambda/4N$ has the same form of the maximally entangled state discussed in Ref. [11].

In this paper we present an overview of some aspects of the enhancement by quantum entanglement in interferometeric devices, and we describe another method for the generation of the desired quantum states. The paper is organized as follows:

In section II we derive the Heisenberg-limited sensitivity and the standard shot-noise limit. Following previous work [19], we then introduce phase estimation with quantum entanglement. In the next section (III), we describe the 'Quantum Rosetta Stone', based on the formal equivalence between the Mach-Zehnder interferometer, atomic clocks, and a generic quatum logic circuit. In section IV we discuss three different ways of achieving the Heisenberg limit sensitivity. A brief description of quantum interferometric lithography and the desired quantum state of light is given in section V. In section VI we discuss quantum state preparation with projective measurements and its application to Heisenberg-limited interferometry.

II. THE HEISENBERG UNCERTAINTY PRINCIPLE AND PARAMETER ESTIMATION

In this section we briefly derive the measurement-sensitivity limits due to Heisenberg's uncertainty principle, and how, in general, quantum entanglement can be used to achieve this sensitivity. There are several stages in the procedure where quantum entanglement can be exploited, both in the state preparation and in the detection. First, we derive the Heisenberg limit, then we give the classical shot-noise limit, and we conclude this section with entanglement enhanced parameter estimation.

Suppose we have a $(2N+1)$-level system. Furthermore, we use the angular momentum representation to find the minimum uncertainty ΔL in an observable L that is a dual to the angular momentum operator J_z. That is, L and J_z obey a Heisenberg uncertainty relation:

$$\Delta L\, \Delta J_z \geq \frac{\hbar}{2} \, . \tag{1}$$

When we want minimum uncertainty in L (minimize ΔL), we need to maximize the uncertainty in J_z (maximize ΔJ_z). Given the eigenstates $\{|m\rangle\}_{m=-N}^{+N}$ of J_z: $J_z|m\rangle = m|m\rangle$, maximum uncertainty in J_z implies the state $|\psi\rangle = N^{-\frac{1}{2}} \sum_{m=-N}^{+N} e^{i\phi_m}|m\rangle$. The variance in J_z is then given by

$$(\Delta J_z)^2 = \langle\psi|J_z^2|\psi\rangle - \langle\psi|J_z|\psi\rangle^2 = \frac{1}{3}\left(N^2+N\right) \, . \tag{2}$$

It immediately follows that the leading term in ΔJ_z scales with N. Using the equality sign in Eq. (1), i.e., minimum uncertainty, and the expression for ΔJ_z, we find that $\Delta L \propto 1/N$.

This result gives the spread of measurement outcomes of an observable L in a $(2N+1)$-level system. However, it is not yet cast in the language of standard parameter estimation. The next question is therefore how to achieve this *Heisenberg-limited* sensitivity when we wish to estimate the value of a parameter φ in N trials.

Classically, the shot-noise limit acording to estimation theory is given by $\Delta\varphi = N^{-\frac{1}{2}}$. We give a short derivation of this value and generalize it to the quantum mechanical case. Consider an ensemble of N two-level systems in the state $|\varphi\rangle = (|0\rangle + e^{i\varphi}|1\rangle)/\sqrt{2}$, where $|0\rangle$ and $|1\rangle$ denote the two levels. If $\hat{A} = |0\rangle\langle 1| + |1\rangle\langle 0|$, then the expectation value of $\hat{A}$ is given by

$$\langle\varphi|\hat{A}|\varphi\rangle = \cos\varphi \, . \tag{3}$$

When we repeat this experiment N times, we obtain

$${}_N\langle\varphi|\ldots{}_1\langle\varphi| \left(\bigoplus_{k=1}^{N} \hat{A}^{(k)}\right) |\varphi\rangle_1 \ldots |\varphi\rangle_N = N\cos\varphi \, . \tag{4}$$

Furthermore, it follows from the definition of $\hat{A}$ that $\hat{A}^2 = \mathbb{1}$ on the relevant subspace, and the variance of $\hat{A}$ given N samples is readily computed to be $(\Delta A)^2 = N(1-\cos^2\varphi) = N\sin^2\varphi$. According to estimation theory [2], we have

$$\Delta\varphi = \frac{\Delta A}{|d\langle\hat{A}\rangle/d\varphi|} = \frac{1}{\sqrt{N}} \, . \tag{5}$$

This is the standard variance in the parameter φ after N trials. In other words, the uncertainty in the phase is inversely proportional to the square root of the number of trials. This is the shot-noise limit.

Quantum entanglement can improve the sesitivity of this procedure by a factor of $\sqrt{N}$. We will first employ a path-entangled state

$$|\varphi_N\rangle \equiv |N,0\rangle + e^{iN\varphi}|0,N\rangle \, , \tag{6}$$

where $|N\rangle$ is a collective state of N qubits. The relative phase $e^{iN\varphi}$ can be obtained by a unitary evolution of one of the modes of $|\varphi_N\rangle$. When we measure the observable $\hat{A}_N = |0,N\rangle\langle N,0| + |N,0\rangle\langle 0,N|$ we have

$$\langle\varphi_N|\hat{A}_N|\varphi_N\rangle = \cos N\varphi \, . \tag{7}$$

Again, $\hat{A}_N^2 = \mathbb{1}$ on the relevant subspace, and $(\Delta A_N)^2 = 1 - \cos^2 N\varphi = \sin^2 N\varphi$. Using Eq. (5) again, we obtain the so-called Heisenberg limit to the minimal detectable phase:

$$\Delta\varphi_H = \frac{\Delta A_N}{|d\langle\hat{A}_N\rangle/d\varphi|} = \frac{1}{N} \, . \tag{8}$$

The precision in φ is increased by a factor $\sqrt{N}$ over the standard noise limit. As shown in Bollinger *et al.* [11], Eq. (8) is the optimal accuracy permitted by the Heisenberg uncertainty principle. In quantum lithography, one exploits the $\cos(N\varphi)$ behaviour, exhibited by Eq. (7), to print closely spaced lines on a suitable substrate [14]. Gyroscopy and entanglement-enhanced frequency measurements [12] exploit the $\sqrt{N}$ increased precision. The physical interpretations of A_N and the phase φ might differ in the different protocols. In the following two sections we present three distinct physical representations of this construction. We call this the *quantum Rosetta stone.*

III. QUANTUM ROSETTA STONE

In this section we discuss the formal equivalence between the Mach-Zehnder interferometer, the Ramsey spectroscope, and a generic quantum gate.

In a Mach-Zehnder interferometer, the input light field is devided into two different paths by a beam splitter, and recombined by another beam splitter. The phase difference between the two paths is then measured by balanced detection of the two output modes (see Fig. 1a). With a coherent laser field as the input the phase sensitivity is given by the shot noise limit $N^{-\frac{1}{2}}$, where N is the average number of photons passing though the interferometer during measurement time. When the number of photons is exactly known (i.e., the input is a Fock state $|N\rangle$), the phase sensitivity is still given by $N^{-\frac{1}{2}}$, indicating that the photon counting noise does not originate from the intensity fluctuations of the input beam [20,12].

By contrast, in a Ramsey spectroscope, atoms are put in a superposition of the ground state and an excited state with a $\pi/2$-pulse (Fig. 1b). After moving through a cavity or another medium, a second $\pi/2$-pulse is applied to the atom, and depending on the relative phase shift obtained by the excited state, the outgoing atom is either in the ground or the excited state. This is essentially an atomic clock.

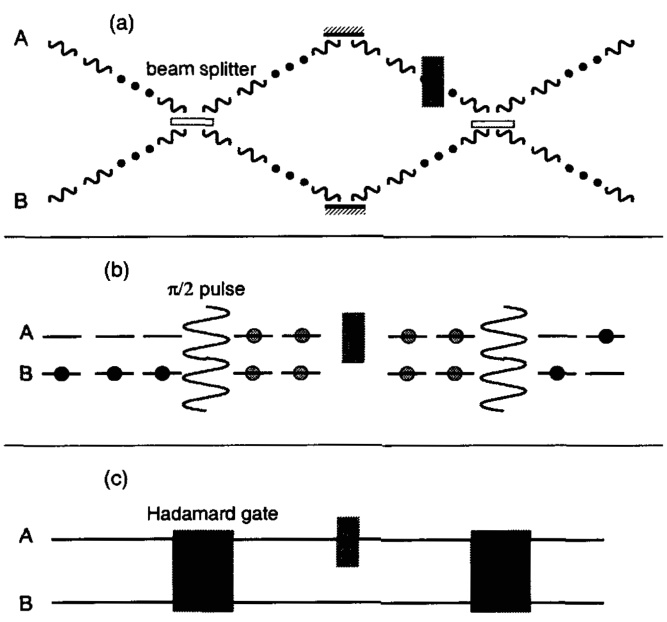

FIG. 1. The quauntum Rosetta stone. (a) An optical Mach-Zenhder interferometer. (b) Ramsey atomic clock. (c) A quantum logic gate representing the equivalent physical proccess.

Let denote $\hat{a}^\dagger$, $\hat{b}^\dagger$ for the two input mode operators in Fig. 1(a). In the Schwinger representation, the common eigenstates of $\hat{J}^2$ and $\hat{J}_z$ are the two-mode Fock states $|j,m\rangle = |j+m\rangle_A |j-m\rangle_B$ where

$$\hat{J}_x = (\hat{a}^\dagger\hat{b} + \hat{b}\hat{a}^\dagger)/2 \quad \hat{J}_y = -i(\hat{a}^\dagger\hat{b} - \hat{b}\hat{a}^\dagger)/2$$
$$\hat{J}_z = (\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})/2 \quad \hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2 \tag{9}$$

The role of the interferometer can be described by the rotation of the angular momemtum vector, where $\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b} = N = 2j$, and the 50/50 beam splitters and the phaser shift are corresponding to the operators $e^{i\pi\hat{J}_x/2}$ and $e^{i\phi\hat{J}_z}$, respectively. Such a formalism is analogous to therotation of the Bloch vector describing a two level atomic system.

A third system is given by a qubit that undergoes a Hadamard transform H, then picks up a relative phase and is then transformed back with a second Hadamard transformation (Fig. 1c). We call the formal analogy between these three systems the ***quantum Rosetta stone.*** In every protocol, the initial state is transformed by a discrete Fourier transform (beam splitter, $\pi/2$-pulse or Hadamard), then picks up a relative phase, and is transformed back again. As a consequence, the phase shift (which is hard to measure directly) is applied to the transformed basis. The result is a bitflip in the initial, ***computational***, basis $\{|0\rangle, |1\rangle\}$, and this is readily measured.

These schemes can be generalized from measuring a simple phase shift to evaluating the action of a unitary transformation U_f associated with a complicated function f on multiple qubits. Such an evaluation is also known as a quantum computation. According to our Rosetta stone, the concept of quantum computers is therefore to exploit quantum interference in obtaining the outcome of a computation of f. In this light, a quantum computer is nothing but a (complicated) multiparticle quantum interferometer [21].

IV. QUANTUM ENHANCEMENT IN PHASE MEASUREMENTS

There have been various proposals for achieving the Heisenberg limit sensitivity, corresponding to different physical realizations of the state $|\varphi_N\rangle$ and observable $\hat{A}_N$ in Eq. (7). Here, we discuss three different approaches, categorized according to the different quantum states. We distinguish Yurke states, dual Fock states, and maximally path-entangled states.

A. Yurke states

By utilizing the $su(2)$ algebra of spin angular momentum, Yurke ***et al.*** have shown that, with a suitably correlated input state, the phase sensitivity can be improved to $1/N$ [4,5]. For spin-1/2 fermions, the entangled input state (which we call the 'Yurke state') is given by

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[\left|j=\frac{N}{2},m=\frac{1}{2}\right\rangle + \left|j=\frac{N}{2},m=-\frac{1}{2}\right\rangle\right]$$
$$= \frac{1}{\sqrt{2}}\left[\left|\frac{N+1}{2},\frac{N-1}{2}\right\rangle_{AB} + \left|\frac{N-1}{2},\frac{N+1}{2}\right\rangle_{AB}\right], \quad (10)$$

where the notion of $|j,m\rangle$ follows the definition given in Eq. (9) and the subscripts AB denote the two input modes. For bosons, a similar input state, namely $|j=N/2,m=0\rangle+|j=N/2,m=1\rangle$, has been proposed [5]. Although the input state of Eq. (10) was proposed for spin-1/2 fermions, the same state with bosons also yields the phase sensitivity of the order of $1/N$ [12]. Furthermore, it has been shown that the Heisenberg limited sensitivity can be achieved by using so-called the minimum uncertainty state or the intelligent state. The minimum uncertainty state is defined as $\Delta J_x \Delta J_y = |\langle J_z\rangle|/2$, and such a state with $\Delta J_y \to 0$ can yield the Heisenberg limited sensitivity under certain conditions [22,23].

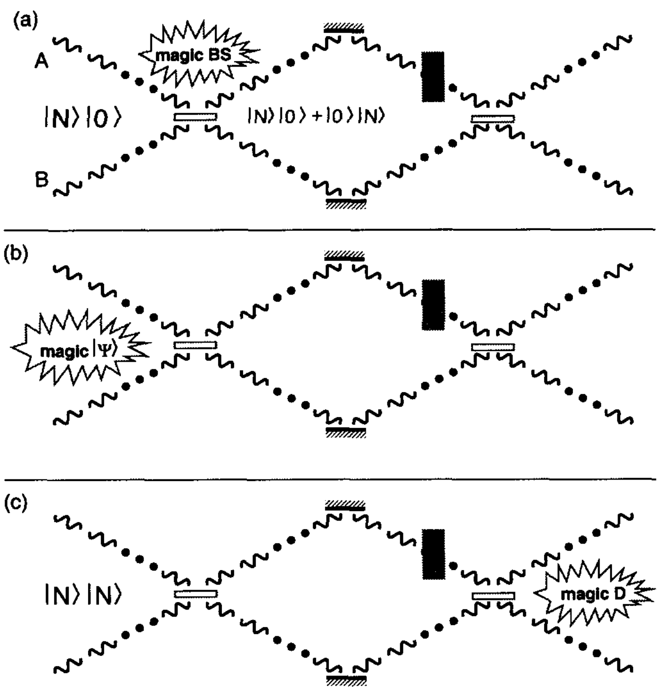

FIG. 2. Three catagories for achieving the Heisenberg limited phase sensitivity. Emphasis on the distinctive features are termed as "magic". (a) Correlated input state, (b) dual Fock-state input, (c) maximally correlated state.

B. Dual Fock states

In order to achieve Heisenberg-limited sensitivity, Holland and Burnett proposed the use of so-called *dual Fock states* $\sum_N c_N|N\rangle_A \otimes |N\rangle_B$ for two input modes A and B of the Mach-Zehnder interferometer [8]. Such a state can be generated, for example, by degenerate parametric down conversion, or by optical parametric oscillation [24].

To obtain increased sensitivity with dual Fock states, some special detection scheme is needed. In a conventional Mach-Zehnder interferometer only the difference of the number of photons at the output is measured. Here, one measures both the sum and the difference of the photon number in the two output modes. The sum contains information about the total photon number, and the difference contains information about the phase shift. Information about the total photon number then allows for post-processing the information about the photon-number difference.

Similarly, in atom interferometers, measurements are performed by counting the number of atoms in a specific internal state using fluorescence. For the schemes using dual Fock-state input, an additional measurement is required since the average in the intensity difference of the two output ports does not contain information about the phase shift. A combination of a direct measurement of the variance of the difference current and a data-processing method based on Bayesian analysis was proposed by Kim *et al.* [24]. For atom interferometers, a quantum nondemolition measurement is required to give the total number of atoms [10]. In a similar context, Yamamoto and coworkers devised an atom interferometry scheme that uses a $\pi/2$ pulse for the readout of the input state correlation [9].

Due to its simple form, the dual Fock-state approach may shed a new light on Heisenberg-limited interferometry. In particular, exploiting the fact that atoms in a Bose-Einstein condensate can be represented by Fock states, Bouyer and Kasevich have shown that the quantum noise in atom interferometry using dual Bose-Einstein consensates can also be reduced to the Heisenberg limit [10].

C. Maximally path-entangled states

The third, and last, category of states is given by the maximally path-entangled states.

There have been proposals to overcome the standard shot noise limit in frequency standard and spectroscopy by using spin-squeezed states [25–29]. However, it has been demonstrated by Wineland and coworkers that the optimal frequency measurement can be achieved by using *maximally correlated states*, which has the form as [11]

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[\left|j=\frac{N}{2},m=\frac{N}{2}\right\rangle + \left|j=\frac{N}{2},m=-\frac{N}{2}\right\rangle\right]$$
$$= \frac{1}{\sqrt{2}}\left[|N,0\rangle_{AB} + |0,N\rangle_{AB}\right]. \quad (11)$$

Although in frequency measurement partially entangled states with high degree of symmetry was later shown to have a better resolution in presence of decoherence

[30,31], such a maximally correlated state is of great interest in optical interferometers.

All the interferometer schemes using the correlated input states, or the dual Fock-state input, show the phase sensitivity approches to $1/N$, only asymptotically, However, using the maximally correlated states of Eq. (11), the phase sensitivity is equal to $1/N$, even for a small N. One distinctive feature compared to the other schemes described above is that it is the desired quantum state after the first beam splitter in the Mach-Zehnder interferometer, not the input state. In that the desired input state is described as the inverse beam-splitter operation to the state of Eq. (11).

V. QUANTUM LITHOGRAPHY AND PROJECTIVE MEASUREMENTS

Quantum correlations can also be applied to optical lithography. In recent work it has been shown that the Rayleigh diffraction limit in optical lithography can be circumvented by the use of path-entangled photon number states [13,14]. The desired N-photon path-entangled state, for N-fold resolution enhancement, is again of the form given in Eq. (6).

Consider the simple case of a two-photon Fock state $|1\rangle_A|1\rangle_B$, which is a natural component of a spontaneous parametric down-conversion event. After passing through a 50/50 beam splitter, it becomes an entangled number state of the form $|2\rangle_A|0\rangle_B + |0\rangle_A|2\rangle_B$. Interference suppresses the probability amplitude of $|1\rangle_A|1\rangle_B$. According to quantum mechanics, it is not possible to tell whether both photons took path A or B after the beam splitter.

When parametrizing the position x on the surface by $\varphi = \pi x/\lambda$, the deposition rate of the two photons onto the substrate becomes $1 + \cos 2\varphi$, which has twice better resolution ($\lambda/8$) than that of single-photon absorption, $1 + \cos\varphi$, or that of uncorrelated two-photon absorption, $(1 + \cos\varphi)^2$. For N-photon path-entangled state of Eq. (11), we obtain the deposition rate $1 + \cos N\varphi$, corresponding to a resolution enhancement of $\lambda/4N$.

It is well known that the two-photon path-entangled state of Eq. (11) can be generated using a Hong-Ou-Mandel (HOM) interferometer [32] and two single-photon input states. A 50/50 beam splitter, however, is not sufficient for producing path-entangled states with a photon number larger than two [33].

In terms of quantum logic gates, the maximally correlated state of the form of Eq. (11) can be made using a Hadamard and a sequence of C-NOT gates. however, building optical C-NOT gates normally requires large nonlinearities. Consequently, in generating such states it is commonly assumed that $\chi^{(3)}$ nonlinear optical components are needed for $N > 2$.

Knill, Laflamme, and Milburn proposed a method for creating probabilistic single-photon quantum logic gates based on teleportation. The only resources for this method are linear optics and projective measurements [34]. Probabilistic quantum logic gates using polarization degrees of freedom have been proposed by Imoto and co-workers, and Franson's team [35,36]. In particular, Pittman, Jacobs, and Franson have experimentally demonstrated polarization-based C-NOT implementations [37]. Using the concept of projective measurements, we have previously demonstrated that the desired path-entangled states can be created when conditioned on the measurement outcome [38,19]. This way, one can avoid the use of large $\chi^{(3)}$ nonlinearities [40].

VI. PROJECTIVE MEASUREMENTS AND HEISENBERG-LIMITED INTERFEROMETRY

The concept of projective measurements can also be applied to the Heisenberg limited interferometry, in which the desired correlation between the two input state can be established directly from the dual Fock state. Let us consider a scheme depicted in Fig. 3. The input modes are transformed by the beam splitters as follows:

$$\begin{aligned} \hat{a}^\dagger &\rightarrow it\hat{a}'^\dagger + r\hat{u}'^\dagger, \\ \hat{b}^\dagger &\rightarrow it\hat{b}'^\dagger + r\hat{v}'^\dagger, \end{aligned} \tag{12}$$

where $i = \sqrt{-1}$, and it and r are the transmission and reflection coefficients given by $t^2 + r^2 = 1$. In our convention, a 50/50 beam splitter, for example, is identified as $t = 1/\sqrt{2}$ and $r = -1/\sqrt{2}$. The mode $\hat{u}'$ and $\hat{v}'$ further pass through an additional 50/50 beam splitter, which is characterized by the transformations

$$\begin{aligned} \hat{u}'^\dagger &\rightarrow (i\hat{d}^\dagger - \hat{c}^\dagger)/\sqrt{2}, \\ \hat{v}'^\dagger &\rightarrow (i\hat{c}^\dagger - \hat{d}^\dagger)/\sqrt{2}. \end{aligned} \tag{13}$$

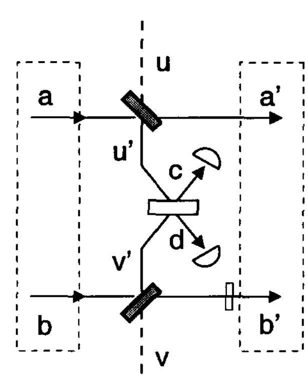

FIG. 3. Generation of the suitable correlation via projective measurements. Mixing the modes $\hat{u}$ and $\hat{v}$ yields a fundamental lack of knowledge about which-path information.

Let us assume the dual Fock state for the input state such as

$$|\Psi_{in}\rangle = |N, N\rangle = \frac{(\hat{a}^\dagger)^N(\hat{b}^\dagger)^N}{N!}|0\rangle. \quad (14)$$

Since we have the beam splitter transformation as

$$(\hat{a}^\dagger)^N \to \sum_k {}_N C_k (\hat{a}'^\dagger)^{N-k}(\hat{u}'^\dagger)^k r^k (it)^{N-k}, \quad (15)$$

after ther first two beam splitters, $|\Psi_{in}\rangle$ transforms as

$$|\Psi_{in}\rangle \to \frac{1}{N!}\sum_{k,l} {}_N C_k \, {}_N C_l (\hat{a}'^\dagger)^{N-k}(\hat{b}'^\dagger)^{N-l} (\hat{u}'^\dagger)^k(\hat{v}'^\dagger)^l r^{k+l}(it)^{2N-k-l}|0\rangle. \quad (16)$$

Now we require the modes $\hat{u}', \hat{v}'$ passing through the 50/50 beam splitter and a signle photon is detected at each detector. Assuming perfect detectors, this restrict the final state after the beam splitter of the form containing only $\hat{c}^\dagger\hat{d}^\dagger$ component. Then, from Eq. (13), we have

$$\begin{aligned}
\hat{u}'^\dagger\hat{v}'^\dagger &\to -i\left[(\hat{c}^\dagger)^2 + (\hat{d}^\dagger)^2\right]/2,\\
(\hat{u}'^\dagger)^2 &\to \left[(\hat{c}^\dagger)^2 - 2i\hat{c}^\dagger\hat{d}^\dagger - (\hat{d}^\dagger)^2\right]/2,\\
(\hat{v}'^\dagger)^2 &\to \left[-(\hat{c}^\dagger)^2 - 2i\hat{c}^\dagger\hat{d}^\dagger + (\hat{d}^\dagger)^2\right]/2, \quad (17)
\end{aligned}$$

and we can see that only $(\hat{u}'^\dagger)^2$ and $(\hat{v}'^\dagger)^2$ terms are selected by the detection of a sinle photon at each detector.

Consequently, we need to select only the terms with $k = 2, l = 0$ or $k = 0, l = 2$ in Eq. (16), so that it gives

$$|\Psi_{in}\rangle \to \frac{1}{N!}\, {}_N C_2 r^2 t^{2N-2} \left[(\hat{a}'^\dagger)^{N-2}(\hat{b}'^\dagger)^N + (\hat{a}'^\dagger)^N(\hat{b}'^\dagger)^{N-2}\right]|0\rangle, \quad (18)$$

where the irrelvant phase factor $(-i)i^{2N-2}$ has been dropped. By replacing $\hat{a}', \hat{b}'$ with $\hat{a}, \hat{b}$, we may write

$$\begin{aligned}
|\Psi_{in}\rangle &\to r^2 t^{2N-2}\frac{1}{N!}\frac{1}{2}(\hat{a}^2 + \hat{b}^2)(\hat{a}^\dagger)^N(\hat{b}^\dagger)^N|0\rangle\\
&= r^2 t^{2N-2}\frac{1}{2}(\hat{a}^2 + \hat{b}^2)|\Psi_{in}\rangle, \quad (19)
\end{aligned}$$

where we have used the relation ${}_N C_2(\hat{a}^\dagger)^{N-2}|0\rangle = (1/2)\hat{a}^2(\hat{a}^\dagger)^N|0\rangle$. Finally, using the input state of $|N, N\rangle$ the output state, conditioned upon coincident count is given by

$$\frac{A}{\sqrt{2}}\left[|N, N-2\rangle + |N-2, N\rangle\right], \quad (20)$$

where $A = \sqrt{N(N-1)/2}\; r^2 t^{2N-2}$, and the maximum probability success is obtained when $r^2 = \frac{1}{N}$ and, $t^2 = \frac{N-1}{N}$.

Now consider that such a state given in Eq. (20) is entering two input port of of the Mach-Zehnder interferometer. An explicit calculation for the phase sensitivity using such an input state is given in Ref. [12]which is approaching to the Heisenberg limit as $O(1/N)$. The generation of desired correlation described in Sec. 3A, can be achieved from the dual Fock state with a probability given by $|A|^2$, which has its asymptotic value of $1/2e^2$.

In atom interferometry a similar technique for the the generation of such a correlation has been proposed [12] by selective measurements on two interfering Bose condensates [41].

VII. SUMMARY

In this paper we readdress the equivalence among the optical Mach-Zehnder interferometer, Ramsey spectroscopy technique, and the generic quantum logic gates. Based on such an equivalence we introduce, so we call, the quantum Rosetta stone. The method of projective measurements applied first in the quantum computing, for example, is found to be useful in generation of the desired quantum state of light for quantum interferometric lithography. The dual Fock-state approach to Heisenberg limited interferometry normally accompanied by additional detection schemes. Generation of a suitable correlation from the dual Fock state via projective measurement may be useful by avoiding those complicated signal processings or QND-type input state measurements. One can also envision a single-photon QND device in this paradigm [42]. Many more fascinating insights are expected by the application of the quantum Rosetta stone to quantum metrology and quantum information processings.

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