Effect of Distributed Rod and String Flexibility on Formation Dynamic Stability*

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ABSTRACT

We study the effect of distributed elasticity on the dynamics of an orbiting formation. The paradigm considered in the paper, namely two bodies connected by one spring in formation with a separated free-flying body, represents a prototype system for a variety of possible problems. Conclusions are derived on the system's stability, as well as on the system performance under closed-loop control.

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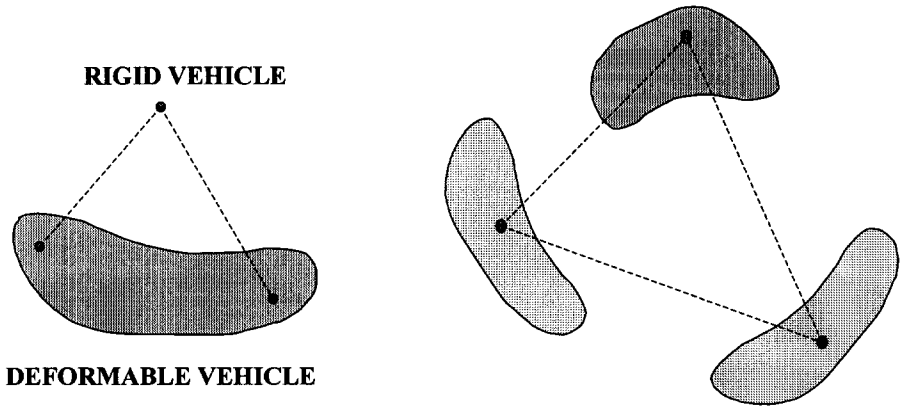

Figure 1: Formation Configurations considered in this paper.

1 Introduction

In this paper we try to answer the following question: what is the effect of the assumption that one of the bodies of an orbiting formation is deformable? Several research papers considering the dynamics and control of envisioned formation flying missions assume that all the elements of the formation can be modeled as rigid bodies. Most of the applications considered so far are using essentially rigid spacecraft (no flexibility involved, except for what may arise from large heatshields) for space interferometry (Terrestrial Planet Finder, StarLight). In parallel, formation flying applications using tethers have also been proposed in the past (see[2]) and more recently (SPECS). In the case of tethers, a long deformable element is present between one or more bodies of the formation. One may assume that, in a not too far future, formation flying missions will be proposed which will consider deformable surfaces such as large reflective shell-like elements, or combinations of tether-connected spacecraft with free-flying spacecraft. In this paper, we would like to consider the problem of formation dynamics and control when part of the formation contains distributed flexibility. This may be the case of a tethered connected pair of vehicles, which are required to maintain a formation with a separated free-flying spacecraft in a different orbit. The model considered in this paper can be thought as a prototype model because it also may provide basic information on the dynamics and control behavior of formations in which a portion of its elements are part of a deformable body (inflatable reflector, flexible antenna). Some of these concepts are depicted in ??.

Previous related work on the virtual truss associated with a formation, which will be used later on in this paper, can be found in[2] and,[3] and[1] represents one of the first reports of a study on the effects of interaction between orbital dynamics and spacecraft flexibility.

First, we describe the assumptions of the model. Second, we develop the equations of motion of the prototype system. Third, we generalize the equations to the case of relative motions. Fourth, we study the stability using the concept of the virtual truss associated with a formation. Finally, the conclusions summarize the results of the paper, and outline directions of future research.

2 Assumptions of Model

The following assumptions are used in this paper:

- The formation is composed of three bodies represented by point masses.

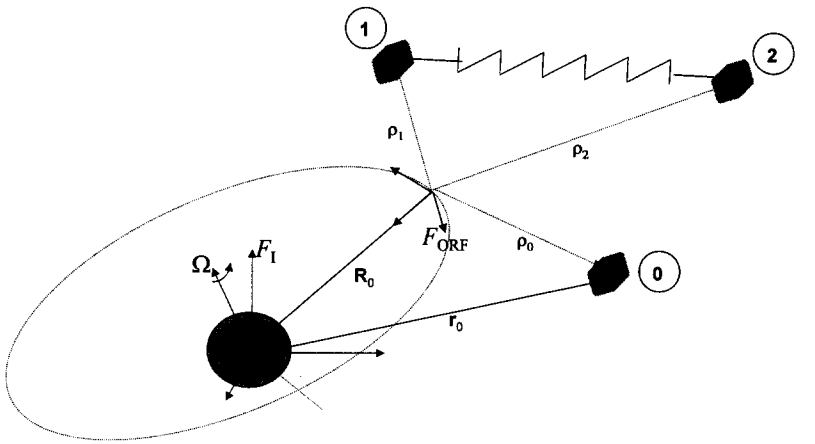

Figure 2: One elastic dumbell orbiting in formation with a rigid spacecraft.

- Two of the bodies are connected by one viscoelastic spring (dumbbel, tether).
- The orbit is circular.
- The formation dynamics is described (and numerically integrated) with respect to the Orbiting Reference Frame, to be described next.

We follow Figure 2. The motion of the system is described with respect to a local vertical-local horizontal (LV-LH) orbiting reference frame $(x, y, z) = \mathcal{F}_{ORF}$ of origin O_{ORF} which rotates with mean motion Ω and orbital radius R_0. A general type of orbit can be accommodated in the model, as the orbital geometry at the initial time is defined in terms of its six orbital elements, and the orbital dynamics equation for point O_{ORF} is propagated forward in time under the influence of the gravitational field of the primary (Earth for LEO, Sun for Deep Space applications) and of the Earth as third body perturbation effect. The origin of this frame coincides with the initial position of the center of mass of the system, and the coordinate axes are z along the local vertical, x toward the flight direction, and y in the orbit normal direction. The inertial reference frame $(X, Y, Z) = \mathcal{F}_I$ is geocentric inertial for LEO (X points toward the vernal equinox, Z toward the North Pole, and Y completes the right handed reference frame), and heliocentric inertial for other applications. The orbit of the origin of $\mathcal{F}_{ORF}$ is defined by the six orbital elements a (semimajor axis), e (eccentricity), i (inclination), Ω_l (longitude of ascending node), ϖ (argument of perigee), ν (true anomaly), and time of passage through perigee. The transformation between $\mathcal{F}_{ORF}$ and $\mathcal{F}_I$ is given by $F_{ORF} = RF_I$ with $R = R_3(\varpi + \nu) R_2(i) R_3(\Omega_l)$ where $R_i(\cdot)$ denotes a rotation matrix of $(\cdot)$ around the direction specified by the subscript. It is useful to refer the translational dynamics of body i to the origin of $\mathcal{F}_{ORF}$. From Figure 2, the position vector of a generic structural point with respect to O_{ORF} is denoted by ρ_i, and we have $r_i = R_0 + \rho_i$. We define the state vector as $X = (R_0, \dot{R}_0, \rho_1, q_1, \dot{\rho}_1, \omega_1, ..., \rho_i, q_i, \dot{\rho}_i, \omega_i, ...\rho_N, q_N, \dot{\rho}_N, \omega_N)$ where q_i and ω_i represent the quaternion and angular velocity vector of the i-th spacecraft with respect to $\mathcal{F}_I$.

3 Equations of Motion with respect to the Orbital Reference Frame

The position vectors of the N masses can be written as $(i = 1, .., N)$:

$$\mathbf{R}_i = \hat{\mathbf{R}}_0 + \boldsymbol{\rho}_i \tag{1}$$

where $\rho_i = \begin{pmatrix} x_i & y_i & z_i \end{pmatrix} \mathcal{F}_{ORF}$, $\hat{R}_0 = R_0 \mathcal{F}_{ORF} = R_0 \mathbf{b}_1$, and $\dot{\mathcal{F}}^T_{ORF} = \mathcal{F}^T_{ORF} \tilde{\Omega}$, $\Omega = \Omega \mathbf{b}_3$. The velocity vector becomes:

$$\dot{\mathbf{R}}_i = \begin{pmatrix} \dot{x}_i - \Omega y_i & \dot{y}_i + \Omega(R_0 + x_i) & \dot{z}_i \end{pmatrix} \mathcal{F}_{ORF} \tag{2}$$

$$\ddot{\mathbf{R}}_i = \ddot{\mathbf{R}}_0 + \ddot{\boldsymbol{\rho}}_i + \boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \boldsymbol{\rho}_i + 2\boldsymbol{\Omega} \times \dot{\boldsymbol{\rho}}_i \tag{3}$$

The kinetic energy of the system can then be written as

$$T = \frac{1}{2} \sum_{i=1}^{N} m_i \left\{ \dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2 + \Omega^2 [y_i^2 + (R_0 + x_i)^2] + 2\Omega [\dot{y}_i (R_0 + x_i) - \dot{x}_i y_i] \right\} \tag{4}$$

The total gravitational potential energy of the system is ($\mu = GM$):

$$V_g = -\mu \sum_{i=1}^{N} \frac{m_i}{|\mathbf{R}_i|} \tag{5}$$

which, with the usual approximation:

$$\begin{aligned} |\mathbf{R}_i|^{-1} &= [(R_0 + x_i)^2 + y_i^2 + z_i^2]^{-\frac{1}{2}} \\ &\approx R_0^{-1} (1 - \frac{x_i}{R_0} + \frac{2x_i^2 - y_i^2 - z_i^2}{2R_0^2}) \end{aligned} \tag{6}$$

becomes

$$V_g = -\frac{\mu}{R_0} \sum_{i=1}^{N} m_i (1 - \frac{x_i}{R_0} + \frac{2x_i^2 - y_i^2 - z_i^2}{2R_0^2}) \tag{7}$$

The elastic potential energy of the system of the system of S viscoelastic springs can be written as:

$$V_E = \frac{1}{2} \sum_{i=1}^{S} k_i (\ell_i - \ell_{i0})^2 \tag{8}$$

and the Raileigh dissipation function as

$$R = \frac{1}{2} \sum_{i=1}^{S} c_i \dot{\ell}_i^2 \tag{9}$$

where k and c represent the stiffness and viscous damping constant of the spring, respectively, ℓ_0 is the rest length of the spring, and for the case of one spring present between bodies 2 and 3 we may write

$$\ell = (\ell \cdot \ell)^{\frac{1}{2}} = [(x_3 - x_2)^2 + (y_3 - y_2)^2 + (z_3 - z_2)^2]^{\frac{1}{2}} \tag{10}$$

$$\dot{\ell} = \begin{bmatrix} \dot{x}_3 - \dot{x}_2 - \Omega(y_3 - y_2) & \dot{y}_3 - \dot{y}_2 + \Omega(x_3 - x_2) & \dot{z}_3 - \dot{z}_2 \end{bmatrix} \mathcal{F}_{ORF} \tag{11}$$

and

$$\dot{\ell} = (\dot{\ell} \cdot \dot{\ell})^{\frac{1}{2}} = \left\{ [\dot{x}_3 - \dot{x}_2 - \Omega(y_3 - y_2)]^2 + [\dot{y}_3 - \dot{y}_2 + \Omega(x_3 - x_2)]^2 + (\dot{z}_3 - \dot{z}_2)^2 \right\}^{\frac{1}{2}} \tag{12}$$

The equations of motion can be derived from the well-known form

$$\frac{d}{dt} (\frac{\partial T}{\partial \dot{q}_i}) - \frac{\partial (T - V_g - V_E)}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} = Q_i \tag{13}$$

where Q_i are the generalized forces, and $q = \begin{pmatrix} x_1 & y_1 & z_1 & \dots & x_N & y_N & z_N \end{pmatrix}$ represents the generalized coordinates in matrix form.

For the system with three masses and one spring, the equations of motion become

$$\ddot{x}_1 - 2\Omega\dot{y}_1 - 3\Omega^2 x_1 = \frac{Q_{x_1}}{m_1} \tag{14}$$

$$\ddot{y}_1 + 2\Omega\dot{x}_1 = \frac{Q_{y_1}}{m_1} \tag{15}$$

$$\ddot{z}_1 + \Omega^2 z_1 = \frac{Q_{z_1}}{m_1} \tag{16}$$

$$\ddot{x}_2 - 2\Omega\dot{y}_2 - 3\Omega^2 x_2 + c(\dot{x}_2 - \dot{x}_3) + \gamma(x_2 - x_3) = \frac{Q_{x_2}}{m_2} \tag{17}$$

$$\ddot{y}_2 + 2\Omega\dot{x}_2 + c(\dot{y}_2 - \dot{y}_3) + \gamma(y_2 - y_3) = \frac{Q_{y_1}}{m_2} \tag{18}$$

$$\ddot{z}_2 + \Omega^2 z_2 + c(\dot{z}_2 - \dot{z}_3) + \gamma(z_2 - z_3) = \frac{Q_{z_2}}{m_2} \tag{19}$$

$$\ddot{x}_3 - 2\Omega\dot{y}_3 - 3\Omega^2 x_3 + c(\dot{x}_3 - \dot{x}_2) + \gamma(x_3 - x_2) = \frac{Q_{x_3}}{m_3} \tag{20}$$

$$\ddot{y}_3 + 2\Omega\dot{x}_3 + c(\dot{y}_3 - \dot{y}_2) + \gamma(y_3 - y_2) = \frac{Q_{y_3}}{m_3} \tag{21}$$

$$\ddot{z}_3 + \Omega^2 z_3 + c(\dot{z}_3 - \dot{z}_2) + \gamma(z_3 - z_2) = \frac{Q_{z_3}}{m_3} \tag{22}$$

where the generalized stiffness parameter (not restricted to small deformations) is

$$\gamma = \frac{EA}{\ell}\varepsilon \tag{23}$$

and

$$\varepsilon = \frac{\ell}{\ell_0} - 1 \tag{24}$$

is the generalized longitudinal strain.

For the system with four masses and two springs, the equations of motion become

$$\ddot{x}_1 - 2\Omega\dot{y}_1 - 3\Omega^2 x_1 = \frac{Q_{x_1}}{m_1} \tag{25}$$

$$\ddot{y}_1 + 2\Omega\dot{x}_1 = \frac{Q_{y_1}}{m_1} \tag{26}$$

$$\ddot{z}_1 + \Omega^2 z_1 = \frac{Q_{z_1}}{m_1} \tag{27}$$

$$\ddot{x}_2 - 2\Omega\dot{y}_2 - 3\Omega^2 x_2 - c_1(\dot{x}_3 - \dot{x}_2) - \gamma_1(x_3 - x_2) = \frac{Q_{x_2}}{m_2} \tag{28}$$

$$\ddot{y}_2 + 2\Omega\dot{x}_2 - c_1(\dot{y}_3 - \dot{y}_2) - \gamma_1(y_3 - y_2) = \frac{Q_{y_1}}{m_2} \tag{29}$$

$$\ddot{z}_2 + \Omega^2 z_2 - c_1(\dot{z}_3 - \dot{z}_2) - \gamma_1(z_3 - z_2) = \frac{Q_{z_2}}{m_2} \tag{30}$$

$$\ddot{x}_3 - 2\Omega\dot{y}_3 - 3\Omega^2 x_3 + c_1(\dot{x}_3 - \dot{x}_2) + \gamma_1(x_3 - x_2) - c_2(\dot{x}_4 - \dot{x}_3) - \gamma_2(x_4 - x_3) = \frac{Q_{x_3}}{m_3} \tag{31}$$

$$\ddot{y}_3 + 2\Omega\dot{x}_3 + c_1(\dot{y}_3 - \dot{y}_2) + \gamma_1(y_3 - y_2) - c_2(\dot{y}_4 - \dot{y}_3) - \gamma_2(y_4 - y_3) = \frac{Q_{y_3}}{m_3} \tag{32}$$

$$\ddot{z}_3 + \Omega^2 z_3 + c_1(\dot{z}_3 - \dot{z}_2) + \gamma_1(z_3 - z_2) - c_2(\dot{z}_4 - \dot{z}_3) - \gamma_2(z_4 - z_3) = \frac{Q_{z_3}}{m_3} \tag{33}$$

$$\ddot{x}_4 - 2\Omega\dot{y}_4 - 3\Omega^2 x_4 + c_2(\dot{x}_4 - \dot{x}_3) + \gamma_2(x_4 - x_3) = \frac{Q_{x_4}}{m_4} \tag{34}$$

$$\ddot{y}_4 + 2\Omega\dot{x}_4 + c_2(\dot{y}_4 - \dot{y}_3) + \gamma_2(y_4 - y_3) = \frac{Q_{y_4}}{m_4} \tag{35}$$

$$\ddot{z}_4 + \Omega^2 z_4 + c_2(\dot{z}_4 - \dot{z}_3) + \gamma_2(z_4 - z_3) = \frac{Q_{z_4}}{m_4} \tag{36}$$

where $\gamma_1 = \frac{(EA)_1}{\ell_1}\varepsilon_1$, $\gamma_2 = \frac{(EA)_2}{\ell_2}\varepsilon_2$, $\varepsilon_1 = \frac{\ell_1}{\ell_{10}} - 1$, and $\varepsilon_2 = \frac{\ell_2}{\ell_{20}} - 1$.

It is convenient to write the equations of motion in matrix form as follows. Define the generalized coordinate vector S as:

$$\mathsf{S} = \begin{pmatrix} \mathsf{S}_1 & \dots & \mathsf{S}_N \end{pmatrix}^T \tag{37}$$

Then the equations of motion for N masses and S springs with external forces (Q_e) and control forces (Q_c) can be compactly written as:

$$\mathcal{M}\ddot{\mathsf{S}} + (\mathcal{G} + \mathcal{D})\dot{\mathsf{S}} + (\mathcal{C} + \mathcal{K}(\mathcal{S}))\mathsf{S} = \mathcal{Q}_e + \mathcal{Q}_c \tag{38}$$

where

$$\mathcal{M} = \begin{bmatrix} \mathcal{M}_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mathcal{M}_N \end{bmatrix}$$

$$\mathcal{G} = \begin{bmatrix} \mathcal{G}_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mathcal{G}_N \end{bmatrix} \tag{39}$$

$$\mathcal{C} = \begin{bmatrix} \mathcal{C}_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mathcal{C}_N \end{bmatrix} \tag{40}$$

$$\mathcal{Q}_e = \begin{bmatrix} \mathcal{Q}_e^1 \\ \vdots \\ \mathcal{Q}_e^N \end{bmatrix} \tag{41}$$

with

$$\mathcal{M}_i = \begin{bmatrix} m_i & 0 & 0 \\ 0 & m_i & 0 \\ 0 & 0 & m_i \end{bmatrix} \tag{42}$$

$$\mathcal{G}_i = \begin{bmatrix} 0 & -2m_i\Omega & 0 \\ 2m_i\Omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{43}$$

$$\mathcal{C}_i = \begin{bmatrix} -3m_i\Omega^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & m_i\Omega^2 \end{bmatrix} \tag{44}$$

and, denoting the (3,3) identity matrix by 1, for the case of three masses and two springs,

$$\mathcal{D} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & c_1\mathbf{1} & -c_1\mathbf{1} & 0 \\ 0 & -c_1\mathbf{1} & c_1\mathbf{1}+c_2\mathbf{1} & -c_2\mathbf{1} \\ 0 & 0 & -c_2\mathbf{1} & c_2\mathbf{1} \end{bmatrix} \tag{45}$$

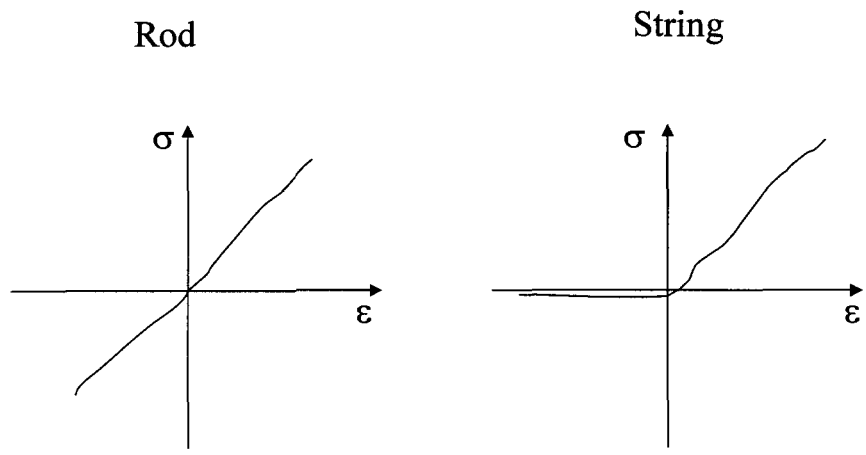

Figure 3: Different constitutive relation between a rod and a string.

$$\mathcal{K} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \gamma_1\mathbf{1} & -\gamma_1\mathbf{1} & 0 \\ 0 & -\gamma_1\mathbf{1} & \gamma_1\mathbf{1}+\gamma_2\mathbf{1} & -\gamma_2\mathbf{1} \\ 0 & 0 & -\gamma_2\mathbf{1} & \gamma_2\mathbf{1} \end{bmatrix} \tag{46}$$

which are obviously of a tri-diagonal form. Note that K is configuration dependent, since γ.depends on the current spring length. Furthermore, as depicted in Figure 3,there is a significant difference between the constitutive response of a rod and that of a string: the rod may respond to compression with a similar intensity with which it responds to tension, whereas a string reacts in a negligible manner in compression. This difference is incorporated in the equations stated above by making

$$rod: \quad \begin{array}{l} \gamma \geq 0 \text{ if } \varepsilon \geq 0 \\ \gamma < 0 \text{ if } \varepsilon < 0 \end{array} \tag{47}$$

$$string: \quad \begin{array}{l} \gamma \geq 0 \text{ if } \varepsilon \geq 0 \\ \gamma = 0 \text{ if } \varepsilon < 0 \end{array} \tag{48}$$

Therefore, the viscoelastic rod model in what follows is representative of a prototype problem of a viscoelastic structure with bilateral springs (orbiting beam), whereas the viscoelastic string model in what follows is representative of a prototype problem of a viscoelastic structure with unilateral springs (tether).

4 Equations of motion in a referential representation

Adopting a Newtonian, rather than Lagrangean, approach, and measuring translational quantities with respect to $\mathcal{F}_{ORF}$,the translational dynamics equations become:

$$m_i\ddot{\boldsymbol{\rho}}_i = \underline{-\mu m_i \left[\frac{\mathbf{R}_i}{|\mathbf{R}_i|^3} - \frac{\mathbf{R}_0}{|\mathbf{R}_0|^3} \right]} - m_i\boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \boldsymbol{\rho}_\mathbf{i} - 2m_\mathbf{i}\boldsymbol{\Omega} \times \dot{\boldsymbol{\rho}}_\mathbf{i} + \mathbf{f}_i + \mathbf{u}_i \tag{49}$$

Assuming that $|\boldsymbol{\rho}_\mathbf{i}| \leq (|\mathbf{R}_0|, |\mathbf{R}_i|)$, let us rewrite the underlined terms (henceforth denoted by G) as follows.

$$\mathcal{G} = \frac{\mathbf{R}_i}{|\mathbf{R}_i|^3} - \frac{\mathbf{R}_0}{|\mathbf{R}_0|^3} = \frac{(\boldsymbol{\rho}_\mathbf{i} + \mathbf{R}_0)}{|\boldsymbol{\rho}_\mathbf{i} + \mathbf{R}_0|^3} - \frac{\mathbf{R}_0}{|\mathbf{R}_0|^3} \tag{50}$$

But

$$
\begin{aligned}
|\boldsymbol{\rho}_\mathbf{i} + \mathbf{R}_0|^{-3} &= [(\boldsymbol{\rho}_\mathbf{i} + \mathbf{R}_0) \cdot (\boldsymbol{\rho}_\mathbf{i} + \mathbf{R}_0)]^{-\frac{3}{2}} \\
&= R_0^{-3} \left(1 + \frac{\rho_\mathbf{i}^2}{R_0^2} + \frac{2\boldsymbol{\rho}_\mathbf{i} \cdot \mathbf{R}_0}{R_0^2}\right)^{-\frac{3}{2}} \\
&\simeq R_0^{-3} \left(1 - \frac{3\boldsymbol{\rho}_\mathbf{i} \cdot \mathbf{R}_0}{R_0^2}\right) \qquad (51)
\end{aligned}
$$

where a binomial expansion has been used in the approximation. Consequently:

$$
\begin{aligned}
\mathcal{G} &\simeq (\boldsymbol{\rho}_\mathbf{i} + \mathbf{R}_0) \cdot R_0^{-3} \left(1 - \frac{3\boldsymbol{\rho}_\mathbf{i} \cdot \mathbf{R}_0}{R_0^2}\right) - \frac{\mathbf{R}_0}{|\mathbf{R}_0|^3} \qquad (52) \\
&= \frac{\boldsymbol{\rho}_\mathbf{i}}{R_0^3} - \frac{3}{R_0^5} (\boldsymbol{\rho}_\mathbf{i} \cdot \mathbf{R}_0) (\boldsymbol{\rho}_\mathbf{i} + \mathbf{R}_0) \qquad (53)
\end{aligned}
$$

Expanding the product using the double vector product equality, and making use of

$$(\boldsymbol{\rho}_\mathbf{i} \cdot \mathbf{R}_0) \boldsymbol{\rho}_\mathbf{i} = \boldsymbol{\rho}_\mathbf{i} \times (\boldsymbol{\rho}_\mathbf{i} + \mathbf{R}_0) + \rho_i^2 \mathbf{R}_0 \qquad (54)$$

$$(\boldsymbol{\rho}_\mathbf{i} \cdot \mathbf{R}_0) \mathbf{R}_0 = \mathbf{R}_0 \times (\mathbf{R}_0 + \boldsymbol{\rho}_\mathbf{i}) + R_0^2 \boldsymbol{\rho}_\mathbf{i} \qquad (55)$$

we obtain

$$\mathcal{G} \simeq \frac{\boldsymbol{\rho}_\mathbf{i}}{r_0^3} - \frac{3}{r_0^5} \left[\mathbf{r}_0 \times (\mathbf{r}_0 + \boldsymbol{\rho}_\mathbf{i}) + r_0^2 \boldsymbol{\rho}_\mathbf{i} + \boldsymbol{\rho}_\mathbf{i} \times (\boldsymbol{\rho}_\mathbf{i} + \mathbf{r}_0) + \rho_i^2 \mathbf{r}_0\right] \qquad (56)$$

and neglecting second order terms in ρ_i, we obtain the final result

$$\mathcal{G} \simeq -\frac{(2R_0^2 \mathbf{1}_3 + 3\mathbf{R}_0 \times (\mathbf{R}_0 \times \mathbf{1}_3))}{R_0^5} \boldsymbol{\rho}_\mathbf{i} \qquad (57)$$

Consequently, the equation of motion of translation, linearized with respect to the orbital motion, becomes:

$$m_i \ddot{\boldsymbol{\rho}}_i + 2m_\mathbf{i} \boldsymbol{\Omega} \times \dot{\boldsymbol{\rho}}_\mathbf{i} + m_i \mathcal{K}_{orb} \boldsymbol{\rho}_\mathbf{i} = \mathbf{f}_i + \mathbf{u}_i \qquad (58)$$

where

$$\mathcal{K}_{orb} = \left[\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{1}_3) - \mu \frac{(2R_0^2 \mathbf{1}_3 + 3\mathbf{R}_0 \times (\mathbf{R}_0 \times \mathbf{1}_3))}{R_0^5}\right] \qquad (59)$$

Similarly, for the $j - th$ body:

$$m_j \ddot{\boldsymbol{\rho}}_j + 2m_\mathbf{j} \boldsymbol{\Omega} \times \dot{\boldsymbol{\rho}}_\mathbf{j} + m_j \mathcal{K}_{orb} \boldsymbol{\rho}_\mathbf{j} = \mathbf{f}_j + \mathbf{u}_j \qquad (60)$$

Subtracting eq.?? from eq. 58, and defining the relative position vector

$$\boldsymbol{\rho}_\mathbf{ji} = \boldsymbol{\rho}_\mathbf{j} - \boldsymbol{\rho}_\mathbf{i} \qquad (61)$$

we obtain the relative equations of motion as:

$$m_j \ddot{\boldsymbol{\rho}}_{ji} + (m_j - m_\mathbf{i}) \ddot{\boldsymbol{\rho}}_\mathbf{i} + 2m_\mathbf{j} \boldsymbol{\Omega} \times \dot{\boldsymbol{\rho}}_\mathbf{ji} + 2(\mathbf{m_j} - \mathbf{m_i}) \boldsymbol{\Omega} \times \dot{\boldsymbol{\rho}}_\mathbf{i} + m_j \mathcal{K}_{orb} \boldsymbol{\rho}_\mathbf{ji} + (\mathbf{m_j} - \mathbf{m_i}) \mathcal{K}_\mathbf{orb} \boldsymbol{\rho}_\mathbf{i} = (\mathbf{f}_j - \mathbf{f}_i) + (\mathbf{u}_j - \mathbf{u}_i) \qquad (62)$$

Now, consider the relative rotational equations. The attitude of the $i - th$ body can be described by the attitude of the dyadic $\mathcal{F}_i$. Then, $\mathcal{F}_i$ may be referenced with respect to: the j-th body, a central body 0, the attitude of the formation center of mass $\mathcal{F}_{ORF}$, the inertial frame $\mathcal{F}_I$. Following our discussion, suppose that $\mathcal{F}_i$ is referenced with respect to $\mathcal{F}_I$. Then, the rotational Euler's equations for the $i - th$ and $j - th$ body are:

$$\mathbf{J}_\mathbf{i} \dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times (\mathbf{J}_i \boldsymbol{\omega}_i + \mathbf{h}_i) = \mathbf{g}_i + \boldsymbol{\tau}_i \qquad (63)$$

$$\mathbf{J}_j \dot{\boldsymbol{\omega}}_j + \boldsymbol{\omega}_j \times (\mathbf{J}_j \boldsymbol{\omega}_j + \mathbf{h}_j) = \mathbf{g}_j + \boldsymbol{\tau}_j \qquad (64)$$

where $\mathbf{J}_i$ and $\mathbf{J}_j$ are the body moment of inertia tensors, $\mathbf{h}_i$ and $\mathbf{h}_j$ represent the contribution of internal angular momentum (from now on, assumed to be zero), $\mathbf{g}_i$ and $\mathbf{g}_j$ are the perturbation torques, and τ_i and τ_j are the control torques. Defining the relative angular velocity

$$\boldsymbol{\omega}_{\mathbf{ji}} = \boldsymbol{\omega}_{\mathbf{j}} - \boldsymbol{\omega}_{\mathbf{i}} \tag{65}$$

and subtracting eq.?? from eq. ??, we obtain

$$\mathbf{J}_j\dot{\boldsymbol{\omega}}_{ji} + (\mathbf{J}_j - \mathbf{J}_\mathbf{i})\,\dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times [(\mathbf{J}_j - \mathbf{J}_\mathbf{i})\,\boldsymbol{\omega}_i] + \underline{\boldsymbol{\omega}_i \times \mathbf{J}_j\boldsymbol{\omega}_{ji} + \boldsymbol{\omega}_{ji} \times (\mathbf{J}_j\boldsymbol{\omega}_j)} = (\mathbf{g}_j - \mathbf{g}_i) + (\boldsymbol{\tau}_j - \boldsymbol{\tau}_i) \tag{66}$$

The underlined terms may be rewritten as

$$\boldsymbol{\omega}_i \times \mathbf{J}_j\boldsymbol{\omega}_{ji} = (\boldsymbol{\omega}_i \times \mathbf{J}_j) \cdot \boldsymbol{\omega}_{ji} \tag{67}$$

$$-\left[(\mathbf{J}_j\boldsymbol{\omega}_j) \times \mathbf{1}_3\right]\boldsymbol{\omega}_{ji} \tag{68}$$

consequently, eq. 66 may be rewritten as:

$$\mathbf{J}_j\dot{\boldsymbol{\omega}}_{ji} + (\mathbf{J}_j - \mathbf{J}_\mathbf{i})\,\dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times [(\mathbf{J}_j - \mathbf{J}_\mathbf{i})\,\boldsymbol{\omega}_i] + [(\boldsymbol{\omega}_i \times \mathbf{J}_j) - [(\mathbf{J}_j\boldsymbol{\omega}_j) \times \mathbf{1}_3]] \cdot \boldsymbol{\omega}_{ji} = (\mathbf{g}_j - \mathbf{g}_i) + (\boldsymbol{\tau}_j - \boldsymbol{\tau}_i) \tag{69}$$

In the rest of this paper we refer the equations of motion of the generic body to the origin of the orbiting reference frame, thereby recovering the C-W equations. A future paper will investigate the case in which the equations of motion are written with respect to another body of the formation.

5 Virtual Truss.

We define a Visibility Array or Graph VA: it describes the instantaneous interconnection between members of the formation, and is described by the link between the pair of members defining that relative position/attitude variable.[3] The virtual truss model is useful to describe the dynamics of a formation as a whole, and is a very useful tool for sensitivity analyses. Self-similar dynamics may be identified (uniform expansion, uniform rotation), as a precursor to more complex models of formation reconfiguration. Changes in formation topology (fault tolerance, addition/deletion of feedback loops from neighboring spacecraft) may be described by means of graph theory-based algorithms. A switching function may also be defined for the configuration, and describes the truth table that realizes the present interconnection between the n members of the formation. This function is useful to implement formation reconfigurations (since it can be used as a functional of the current configuration which becomes rearranged during reconfigurations). The virtual truss is also useful to obtain reduced order models of a formation. Since it invokes the formation modes, techniques such as balanced realizations or modal cost analyses, can be adopted for dynamics and control analyses.

In general, the dynamics are nonlinear, particularly when attitude reconfigurations and external perturbations are included. In our case, we limit our analysis to the design of a suitable feedback control law between members of the formation that takes into account the visibility between members. The components of the visibility array VA are represented by the coefficients a_{ji}, where $a_{ji} = 0$ for total disconnection between i and j, and $a_{ji} = 1$ for total connection. Therefore the a_{ji} matrix is in general non symmetric, and may be fully populated. Clearly, a fractional a_{ji} represents partial connection. A proportional-derivative control law may be thought of as the first order term in the expansion of a more generally nonlinear control law in terms of the perturbations in positions and velocities. For a proportional-derivative-type control, the interconnection terms take the form:

$$\mathcal{Q}_c^i = \sum_{\substack{j=1\\ j\neq i}}^{n} \mathbf{a}_{ji}\mathbf{P}^j\,(\mathsf{S}_j - \mathsf{S}_i) + \sum_{\substack{j=1\\ j\neq i}}^{n} \mathbf{a}_{ji}\mathbf{D}^j\left(\dot{\mathsf{S}}_j - \dot{\mathsf{S}}_i\right) \tag{70}$$

After algebraic manipulation, this expression becomes:

$$\mathcal{Q}_c^i \;=\; \left[\;\mathbf{a}_{i1}\mathbf{P}^1 \quad \mathbf{a}_{i2}\mathbf{P}^2 \quad \cdots \quad \mathbf{a}_{in}\mathbf{P}^n\;\right]\mathsf{S} - \sum_k \left(\mathbf{a}_{ik}\mathbf{P}^k\right)\mathsf{S}_i +$$

$$
\begin{aligned}
& \left[\begin{array}{cccc} \mathbf{a}_{i1}\mathbf{D}^1 & \mathbf{a}_{i2}\mathbf{D}^2 & \cdots & \mathbf{a}_{in}\mathbf{D}^n \end{array}\right] \dot{\mathsf{S}} - \sum_k \left(\mathbf{a}_{ik}\mathbf{D}^k\right) \dot{\mathsf{S}}_i \\
= \; & \mathbf{F}_P^i \mathsf{S} + \mathbf{F}_D^i \dot{\mathsf{S}} - \mathbf{K}_{PD}^i \mathsf{S}_i - \mathbf{G}_{PD}^i \dot{\mathsf{S}}_i
\end{aligned} \tag{71}
$$

Consequently, we obtain:

$$
\mathcal{M}_i \ddot{\mathsf{S}}_i + \left(\mathcal{G}_i + \mathcal{D}_i + \mathbf{G}_{PD}^i\right) \dot{\mathsf{S}}_i + \left(\mathcal{C}_i + \mathcal{K}_i(\mathcal{S}) + \mathbf{K}_{PD}^i\right) \mathsf{S}_i = \mathcal{Q}_e^i + \mathbf{F}_P^i \mathsf{S} + \mathbf{F}_D^i \dot{\mathsf{S}} \tag{72}
$$

Considering all the bodies in the formation, stacking all equations together we obtain:

$$
\begin{aligned}
& \begin{bmatrix} \mathcal{M}_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathcal{M}_N \end{bmatrix} \ddot{\mathsf{S}} + \\
& \left(\begin{bmatrix} \left(\mathcal{G}_1 + \mathbf{G}_{PD}^1\right) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \left(\mathcal{G}_N + \mathbf{G}_{PD}^N\right) \end{bmatrix} + \mathcal{D} - \begin{pmatrix} \mathbf{F}_P^1 \\ \vdots \\ \mathbf{F}_P^n \end{pmatrix} \right) \dot{\mathsf{S}} + \\
& \left(\begin{bmatrix} \left(\mathcal{C}_1 + \mathbf{K}_{PD}^1\right) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \left(\mathcal{C}_N + \mathbf{K}_{PD}^N\right) \end{bmatrix} + \mathcal{K}(\mathcal{S}) - \begin{pmatrix} \mathbf{F}_D^1 \\ \vdots \\ \mathbf{F}_D^n \end{pmatrix} \right) \mathsf{S} \\
= \; & \mathcal{Q}_e
\end{aligned} \tag{73}
$$

and rearranging, we obtain:

$$
\mathcal{M}\ddot{\mathsf{S}} + \left(\mathcal{H} + \mathcal{D} - \mathcal{R}_1\right) \dot{\mathsf{S}} + \left(\mathcal{N} + \mathcal{K}(\mathcal{S}) - \mathcal{R}_2\right) \mathsf{S} = \mathcal{Q}_e \tag{74}
$$

In first order form, the system equations of the large scale system (LSS) are

$$
\begin{aligned}
\dot{X} \; = \; & A(X)X = \\
& \begin{bmatrix} 0 & 1 \\ -\mathcal{M}^{-1}\left(\mathcal{N} + \mathcal{K}(\mathcal{S}) - \mathcal{R}_2\right) & -\mathcal{M}^{-1}\left(\mathcal{H} + \mathcal{D} - \mathcal{R}_1\right) \end{bmatrix} \begin{pmatrix} \mathsf{S} \\ \dot{\mathsf{S}} \end{pmatrix}
\end{aligned} \tag{75}
$$

This equation represents an autonomous nonlinear dynamic system. One may obtain the modal content of the system by setting the characteristic equation equal to zero for a prescribed configuration of interest X_0, such that the dynamical matrix becomes $A(X_0)$. We immediately observe three zero frequencies (analogous to rigid body modes). A root locus analysis may be carried out to identify stability issues. Clearly, for strict LSS system stability, it must be that:

$$
\begin{aligned}
spectrum(\mathcal{H} + \mathcal{D} - \mathcal{R}_1) \; & > \; 0 \\
& \quad \text{and} \\
spectrum(\mathcal{N} + \mathcal{K}(\mathcal{S}) - \mathcal{R}_2) \; & > \; 0
\end{aligned} \tag{76}
$$

or that all the eigenvalues of $A(X_0)$ must have strictly negative real part. A state space realization of the formation may be required to reduce the order of the model for control purposes. This can be achieved with balanced realization techniques. In this case, the input-output controllability and observability characteristics of the formation can be taken into account with modal cost analysis techniques as in.[4] Figures 4, 5, 6, and 7 show four possible combinations of the visibility array on a complex plane, when both the control gains as well as the tether stiffness and damping are changed. One may notice the unsymmetric character of this locus, which occurs when tether slackness arises (for negative values of the stiffness, set to zero for a string). These loci show only the region around the origin of the complex plane. Therefore, an interesting behavior occurs, leading to stability changes, depending on the character of the visibility array.

More general statements may be made for the system's stability. In fact, we may invoke system's stability, subsystem's stability, interconnection stability, and other denominations as well. This is generally known in the literature of large scale systems. The subsystem equations of the virtual truss can be rewritten, in the general nonlinear case, as:

$$
\dot{x}_i = \mathbf{f}_i\left(x_i, t\right) + \mathbf{g}_i\left(x_i, t\right) \tag{77}
$$

$$VA = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

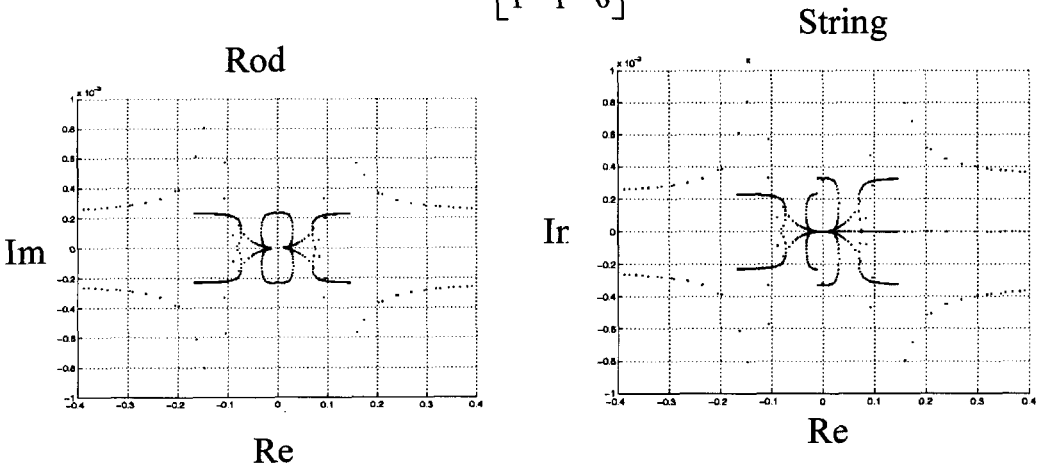

Figure 4: Frequency loci of virtual truss (a). VA represents the current visibility array being used in the loci.

$$VA = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

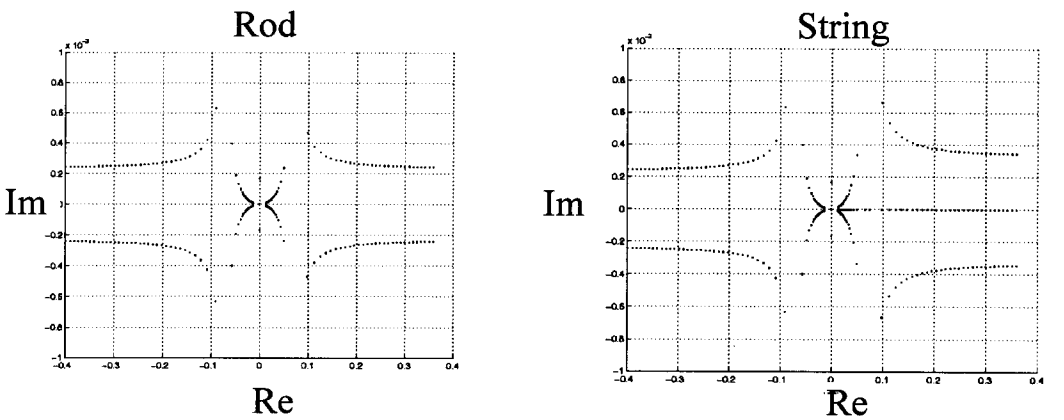

Figure 5: Frequency loci for virtual truss (b).

$$VA = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

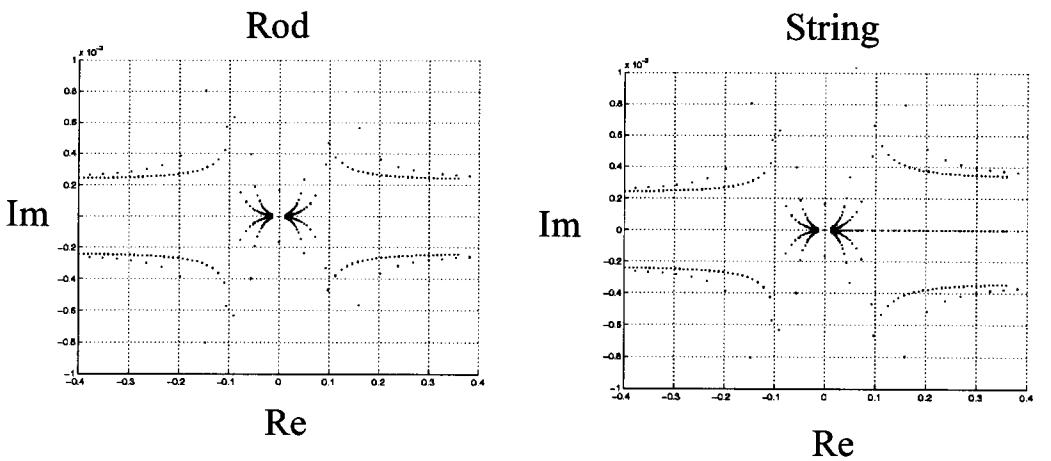

Figure 6: Frequency loci for virtual truss (c).

$$VA = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

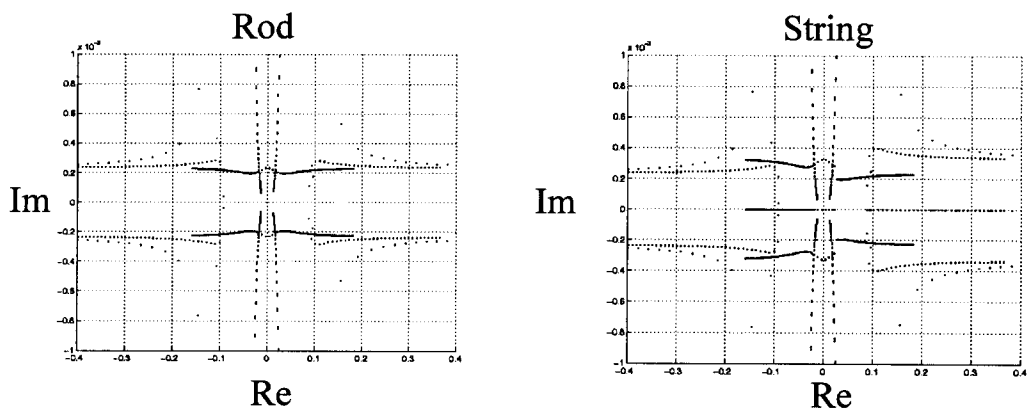

Figure 7: Frequency loci for virtual truss (d).

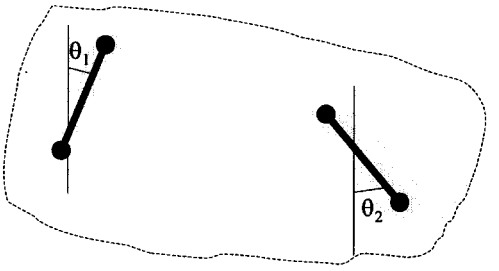

Figure 8: Array of two oscillators.

The second term $\mathbf{g}_i$ is the aggregation or interconnection term. In fact, the equation of each isolated subsystem (ISS) is

$$\dot{x}_i = \mathbf{f}_i\left(x_i, t\right) \tag{78}$$

Two questions arise. What about subsystem stability, i.e. is ISS stable? What about system interconnection stability?

DEFINITION 1. *The equilibrium point X_e is connectively stable if it is stable for all interconnection matrices. The equilibrium point $X_e = 0$ is uniformly asymptotically connectively stable in the large (a.s.i.l.) if there exists a Lyapunov function V(X,t) such that V(X,t) is positive definite, decrescent,*

$$-\dot{V}\left(X, t\right)|_{LSS} > 0 \tag{79}$$

for all X and t (uniform stability) and V(X,t) is radially unbounded.

A question arises: If the system can be decomposed into n ISS subsystems, and we know what the interconnections $\mathbf{g}_i\left(x_i, t\right)$ look like, under what conditions is the original system a.s.i.l.? There exist some theorems[5] with constructive proofs which can be used to assess if the system is a.s.i.l. It is equivalent to finding the conditions under which the subsystems are stable to a degree larger than the strength of the interconnections.

THEOREM 2. *(Araki, 1978) A system is a.s.i.l. if the following conditions hold: 1) for each ISS there is a Lyapunov function $V_i(x_i, t)$ such that*

$$\dot{V}_i\left(x_i, t\right) \leqq -a_i\left[w_i(x_i)\right]^2 \tag{80}$$

$$|\partial V_i/\partial x_i| \leqq w_i(x_i) \tag{81}$$

Where constants $a_i > 0$ and $w_i(x_i)$ is positive definite (indicates degree of ISS stability). 2) For constants $b_{ij} > 0$ (indicating interconnection strength), the interconnection terms are bounded by

$$|\mathbf{g}_i\left(x_i, t\right)| \leqq \sum_j^N b_{ij} w_j(x_j) \tag{82}$$

3) The leading principal minors of the (nxn) matrix E are all >0, where $e_{ii} = a_i - b_{ii}$, and $e_{ij} = - b_{ij}$.

More important for purposes of reconfiguring a formation, is the concept of connective stability, and of controllabil.ity and observability under structural perturbations. Both concepts are directly related to the intrinsic graph embedded in the formation. They make use of the interconnection matrix, or visibility graph. The concept of structural perturbations arises in the case of the formation losing or reacquiring a member. In this case, a variable topology can be assigned to the formation, as more intercomnnection terms participate in the stability of the whole system. As an example, consider the situation depicted in Figure 8.The linearized decentralized dynamics of each oscillator about a reference equilibrium angle can be written as:

$$\ddot{\theta}_1 + 2\zeta_1\sigma_1\dot{\theta}_1 + \sigma_1^2\theta_1 = u_1 \tag{83}$$

$$\ddot{\theta}_2 + 2\zeta_2\sigma_2\dot{\theta}_2 + \sigma_2^2\theta_2 = u_2 \tag{84}$$

where we have assumed viscous-type dissipation. The control inputs may be written as:

$$u_1 = -g_{12}(\theta_2 - \theta_1) - h_{12}(\dot{\theta}_2 - \dot{\theta}_1) \tag{85}$$

$$u_2 = -g_{21}(\theta_1 - \theta_2) - h_{21}(\dot{\theta}_1 - \dot{\theta}_2) \tag{86}$$

where, in general, g_{ij} and h_{ij} are elements of the visibility array multiplied by constants representing control gains, and in general $g_{ij} \neq g_{ji}$ and $h_{ij} \neq h_{ji}$. For the homogeneous dynamics, we then obtain:

$$\ddot{\theta}_1 + (2\zeta_1\sigma_1 - h_{12})\,\dot{\theta}_1 + \left(\sigma_1^2 - g_{12}\right)\theta_1 + \underline{g_{12}\theta_2 + h_{12}\dot{\theta}_2} = 0 \tag{87}$$

$$\ddot{\theta}_2 + (2\zeta_2\sigma_2 - h_{21})\,\dot{\theta}_2 + \left(\sigma_2^2 - g_{21}\right)\theta_2 + \underline{g_{21}\theta_1 + h_{21}\dot{\theta}_1} = 0 \tag{88}$$

The underlined terms in these equations represent the interconnection terms. Stability (connective stability) of each subsystem is ensured provided that

$$\begin{aligned} h_{12} &\leq 2\zeta_1\sigma_1 \\ g_{12} &\leq \sigma_1^2 \end{aligned} \tag{89}$$

$$\begin{aligned} h_{21} &\leq 2\zeta_2\sigma_2 \\ g_{21} &\leq \sigma_2^2 \end{aligned} \tag{90}$$

and provided that the effect of the interconnection terms is small and non-perturbing, i.e.:

$$\|h_{ij}\|\,,\|g_{ij}\| \text{ are small} \tag{91}$$

6 Conclusions

In this paper, we have considered the effect of distributed flexibility arising from rod and string models on the dynamics of an orbiting formation. The paradigm considered in the paper, namely two bodies connected by one rod/string-like spring in formation with a separated free-flying body, represents a prototype system for a variety of possible concepts of formation flying involving deformable elements. The dynamic stability of the system is analyzed by parameterizing the loci of natural frequencies of the associated virtual truss when both the control gains as well as the stiffness coefficients are varied. Results show that different stability boundaries occur when unilateral vs. bilateral springs are considered.

Continuation of this work will be in the area of the dynamic stability of formations of entirely deformable spacecraft, and implications for formation control.

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