Soliton Resonance in Bose-Einstein Condensate

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Abstract

A new phenomenon in nonlinear dispersive systems, including a Bose-Einstein Condensate (BEC), has been described. It is based upon a resonance between an externally induced soliton and "eigen-solitons" of the homogeneous cubic Schrödinger equation. There have been shown that a moving source of positive / negative potential induces bright / dark solitons in an attractive / repulsive Bose condensate.

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This paper is devoted to the analysis of special properties of the cubic Schrödinger equation (CSE) [1] driven by a time-dependent external potential. This equation as an approximation for modulated beams in nonlinear optics as well as for the model of superfluids, and in particular, for the Bose-Einstein condensate, has general significance for time dependent dispersive waves. The objective of this work is to investigate interactions between externally induced solitons (via a time-dependent external potential) and "eigen-solitons" characterizing dispersive waves under a time independent potential. It has been expected that the soliton-shaped external energy pumped into the system will induce and intensify "eigen-solitons" in the same way in which the classical resonance works.

Let us start with the mathematical aspect of the problem. A standard form of a normalized one-dimensional cubic Schrödinger equation driven by a time-dependent potential can be presented as

$$iu_t + u_{xx} + \nu|u|^2 u = Vu, \qquad (1)$$

where $u(x,t)$ is the complex state variable, $V(x,t)$ is the external potential generated by a moving source, and ν is the coupling constant. Looking for a resonance, we will assume that the potential V has the same form as the expected solution:

$$V(x,t) = \lambda \mid u(x,t) \mid, \qquad (2)$$

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where λ is a constant. We will look for the asymptotic solution of this equation (1) at $t \to \infty$ in the form of a localized traveling wave

$$u = \mathrm{e}^{i(rx-st)}\, \upsilon(\xi) \tag{3}$$

where r, s are constants to be determined, and υ is a real function of the argument ξ. The coordinate ξ is chosen in the form $\xi = x - Ut$, where U is the speed of the moving potential. Substituting the expression (3) in the Schrodinger equation (1) and choosing $r = U/2$ and $s = U^2/4$ we obtain an ordinary differential equation for the function $\upsilon(\xi)$:

$$\upsilon_{\xi\xi} + \nu\upsilon^3 - \lambda\upsilon^2 = 0 \tag{4}$$

After integration, the left-hand side of the equation (4) is reduced to the following

$$\upsilon_\xi^{\ 2} + \frac{\nu}{2}\upsilon^4 - \frac{2}{3}\lambda\upsilon^3 = 0 \tag{5}$$

Even prior to integration of equation (5), one concludes that the solution has extremum at $\upsilon = 4\lambda/(3\nu)$. Therefore for positive $\upsilon = 4\lambda/(3\nu)$ the signs of the parameters λ and ν must be the same. Moreover, $\upsilon_{\xi\xi} = -16\lambda^3/(27\nu^2)$ at $\upsilon_\xi = 0$. Hence, this solution has maximum if $\lambda > 0$ and minimum if $\lambda < 0$.

The formal solution of the equation (5)

$$\upsilon = \frac{6/\lambda}{\xi^2 + 9\nu/(2\lambda^2)} \tag{6}$$

describes a moving solitary wave induced by the external potential.

There are several interesting properties of the solution. First, the speed U of the soliton as well as its height

$$h = \frac{4}{3}\frac{\lambda}{\nu} \tag{7}$$

are uniquely determined by the moving potential and in particular, by the parameter λ. For $\lambda > 0$, the solution (6) describes bright soliton, with maximum at the extremum point. For $\lambda < 0$ it describes gray or dark solitons. Second, as follows from equation (7), the resonant growth of the bright soliton is restricted since this solution is spread over the whole space. Therefore, the bright soliton is bounded as in a damped classical resonance. However, in contradistinction to the classical resonance, where the external force must possess certain frequencies, here any speed of the external potential will be "picked-up" by the system. It should be noticed that the shape of the induced soliton is different from those of the known "free" bright soliton, which is given by the equation $\upsilon = (\sqrt{2\alpha} / v)\mathrm{sech}(\xi\sqrt{\alpha})$, where $\alpha = r^2 - s$ is the parameter of the model [1]. Third, the dark soliton has the form

$$\upsilon = \frac{6/|\lambda|}{9|v|/(2\lambda^2) - \xi^2} \tag{8}$$

The solution (8) has the minimum at $\xi = 0$, while $\upsilon \to \infty$ at $\xi = \pm 9v / 2\lambda^2$. As follows from equation (8), the dark soliton is not spread over the whole space, and therefore, an external energy is pumped into a localizing space during an infinite period of time. As a result, at $t \to \infty$, the state variable υ becomes unbounded. In this case the underlying model should be reformulated for υ exceeding a certain original value.

Thus, any moving potential whose shape can be approximated by the function (6), induces a forced soliton moving with a constant velocity U, while the height of this soliton is determined by the equation (7). The amplitude of this soliton can be significantly amplified by an appropriate choice of the control parameter λ.

We will demonstrate the soliton resonance in a Bose condensate. BEC of weakly interacting atoms is described by Gross-Pitaevski equation for the order parameter $\psi(x,t)$:

$$i\hbar\psi_t = \left[-\frac{\hbar^2}{2m}\Delta + V + g|\psi|^2\right]\psi \tag{9}$$

where g is the coupling constant, m is the mass of the atom of the condensate and V is the external potential. The coupling constant in (9) is given by $g = 4\pi\hbar^2 a/m$, where a is the s-wave scattering length. We will select the external potential in the form $V = m\Phi$, where Φ is the gravitational potential and normalization conditions for ψ as $\int d\bar{r}\, |\psi(\bar{r})|^2 = N$, where N is the number of atoms. In a one-dimensional case this equation can be written as

$$i\hbar\psi_t + \frac{\hbar^2}{2m}\psi_{xx} - g|\psi|^2\psi = m\Phi(x)\psi \tag{10}$$

After rescaling the variables $\tilde{t} = t / \hbar$ and $\tilde{x} = \sqrt{2m} / \hbar$ the equation (10) becomes identical to the equation (1):

$$i\tilde{\psi}_{\tilde{t}} + \tilde{\psi}_{\tilde{x}\tilde{x}} - g|\tilde{\psi}|^2\tilde{\psi} = \lambda|\tilde{\psi}|\tilde{\psi} \quad (11)$$

where we put $m\Phi = \lambda\,|\,\tilde{\psi}\,|$. The induced soliton density following from equation (11) is:

$$\sqrt{n} = \frac{3\hbar^2/(m\lambda)}{\xi^2 - 9g\hbar^2/(4m\lambda^2)} \quad (12)$$

Following the results, described above, we can identify two types of solitons in BEC. Assuming $\lambda > 0, g < 0$ we obtain a bright soliton of the compression induced by a moving source of a positive potential in attractive BEC. One should recall, that in the case of time independent potential an attractive BEC can have only bright solitons, which are unstable with respect to disturbances in other dimensions [1]. Obviously, for the case of an induced soliton, the problem of instability becomes irrelevant if the external potential is uniformly distributed over the width of the condensate.

When $\lambda < 0, g > 0$, we will obtain a soliton of depression induced by a moving source of a negative potential in repulsive BEC.

As an example we consider an attractive BEC interacting with a linear source, or a long string, of a constant density with the mass anomaly at point x: $\rho(x) = \rho_0 - M\delta(x)$. The string, moving with speed U, creates gravitational field with potential

$$\Phi_l(\xi) = const + \frac{GM}{\sqrt{\xi^2 + \ell^2}}, \quad (13)$$

where G is the Newtonian gravitational constant and l is the shortest distance between the string and the origin of BEC. Using equations (12) and (13), we obtain the value of λ under the condition that the equation $\lambda\sqrt{n} = m\Phi$ is satisfied. The equation for λ will be

$$\lambda = \sqrt{\frac{3GMm\,|\,g\,|}{4l}} \quad (14)$$

Therefore, the bright soliton induced by a mass M moving near the one-dimensional BEC with the speed U on the distance ℓ is

$$\sqrt{n} = \frac{\hbar^2\sqrt{12l/(GMm^3\,|\,g\,|)}}{\xi^2 + 3\hbar^2 l/(4GMm^2)} \quad (15)$$

As follows from this equation, the soliton density distribution depends on the masses m of atoms in the condensate and on the coupling constant g. We can plot the curves (12) and (13) using the relation between the density distribution of soliton $\sqrt{n}$ and the scaled gravitational potential $(m/\lambda)\,\Phi$. The moving mass anomaly $M = 1\mu g$ at distance $l = 1$ meter, given by the solid line in Fig. 1, creates soliton profile shown by diamond line. The profile is obtained for the condensate of ${}^{7}Li$ atoms with the negative effective scattering length $a = -1.45\,nm$. The maximal density of the induced soliton is $n \approx 10^2\ (1/cm^3)$.

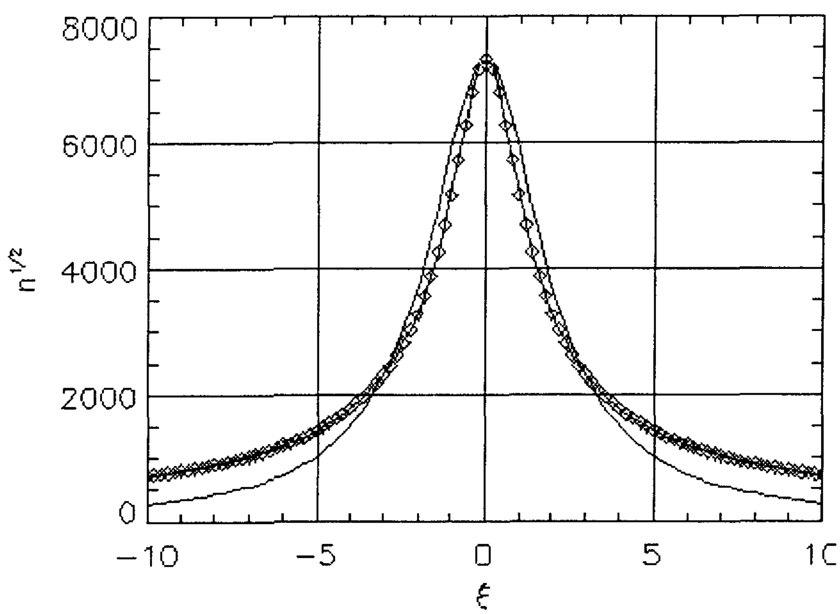

Fig. 1 The *soliton profile* $\sqrt{n}$ *in BEC of* ${}^{7}Li$ *atoms (diamond line) is compared to the scaled gravitational potential* $(m/\lambda)\Phi$ *created by mass anomaly* $M=10^{-7}$ *kg (solid line) moving with constant velocity at the distance* $l = 1$ *cm.*

As follows from Fig.1 the coefficient λ expressed by the equation (14) provides the equality (2) at the top of soliton and at about 80% of its height. In principle, we could find such a distribution of mass anomaly (or to apply an electromagnetic external potential) that would enforce equation (2) exactly. However, we have chosen to demonstrate how a "natural" external potential in the form of a moving point-mass anomaly induces in bright solitons in BEC.

Hence, for a given attractive BEC there has been discovered an optimal moving point-mass anomaly as well as its optimal distance from the BEC such that this motion induces a bright soliton moving with the same speed as the point mass anomaly.

It should be noticed that in this paper we are dealing with asymptotic solutions (1) and (10). Obviously the transitional dynamics of the formation of induced solitons cannot be obtained without a solution of these equations subject to appropriate initial and boundary conditions.

Thus, there has been described a new fundamental phenomenon in nonlinear dispersive systems governed by the cubic Schrödinger equation. It is based upon a resonance between an externally induced soliton and "eigen-soliton" of the homogeneous cubic Schrödinger equation. The analytical form of the forced soliton and its relation to the moving external potential have been established. Special attention was paid to the Bose-Einstein condensate as a nonlinear dispersive system. It has been demonstrated that there are two types of solitons, which can be induced in BEC by external soliton-shaped potential: bright solitons representing solitary waves of compression, and dark solitions representing solitary waves of depression. It has been noticed that the induced solitons are different from the known free solitons not only by the level of their intensity but also by their shapes. The relationships between the type of the soliton, the type of BEC and the sign of the moving external potential have been established.

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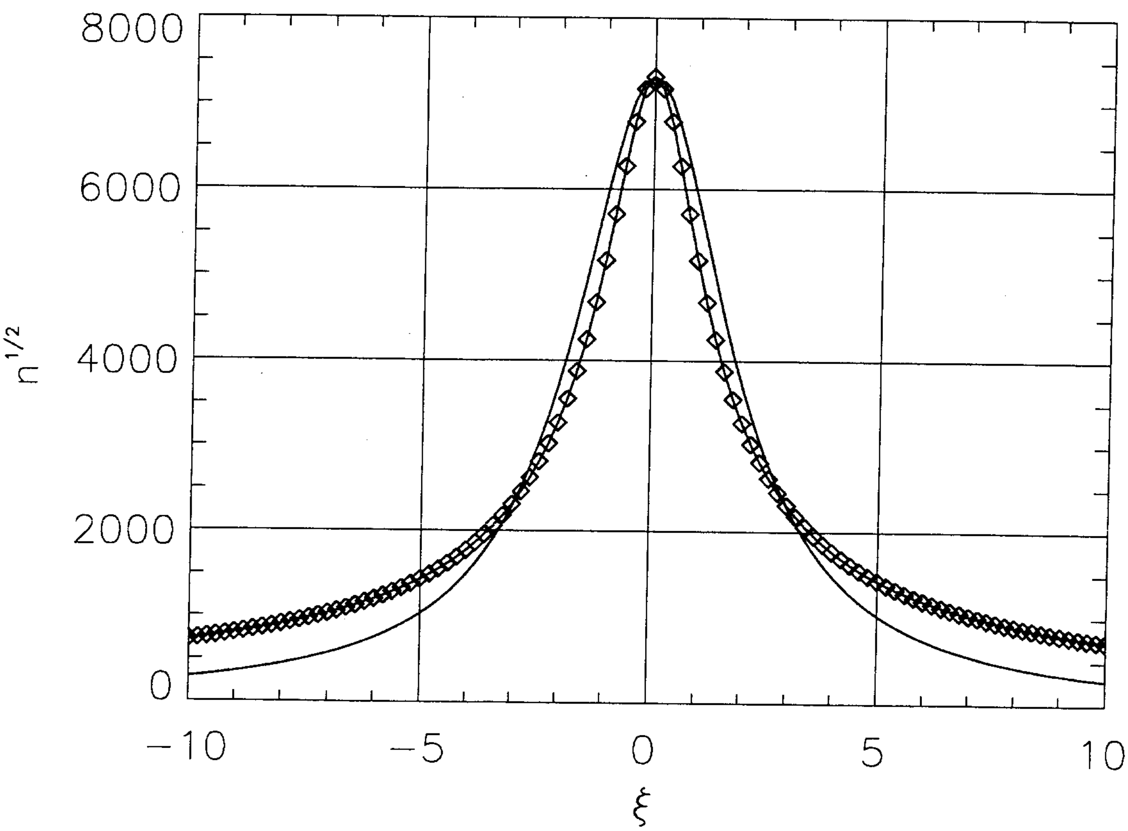
8000
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4000
2000
0
n^1/2
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−5
0
5
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ξ