

FFT-Based Methods for Simulating Flicker FM

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Abstract

Four algorithms for simulating flicker FM phase noise (f^{-3} spectrum) are given, two old and two new. Their Allan deviation and mean square time interval error (MSTIE) are examined. The MSTIE shows that one of the old algorithms has deficient long-term phase deviations on average. The Allan deviation does not reveal this deficiency.

1 Introduction

By a flicker FM model we mean a stochastic process that has (in a sense that can be made precise) a spectral density that is asymptotically equal to const/f^3 as $f \rightarrow 0$. Many quartz oscillators show flicker FM phase noise over wide intervals of Fourier frequency, as evidenced by Allan deviation plots that are approximately flat over two or more decades of averaging time. Therefore, simulations of systems containing quartz oscillators need to include flicker FM generators. This is not a simple matter; for example, a single low-order filter applied to white noise cannot stay close to an odd spectral power over a wide enough frequency range. Following are three classes of existing flicker FM generation algorithms. They all have running time of order $N \log N$, where N is the number of points to be generated.

- Barnes–Jarvis generators [1]. White noise is applied to a ladder of first-order filters constructed to have an approximate $f^{-1/2}$ response over a frequency range whose low end depends on the number N of points to be generated. The result is a stationary process with approximate spectrum f^{-1} over this range. One can obtain f^{-3} noise by a cumulative sum operation on the f^{-1} noise. Because these algorithms generate the output sequentially, they take little memory.
- Discrete spectrum (DS) generators. Complex-valued Hermitian white noise is generated in the frequency domain (the Fourier transform of time-domain white noise), multiplied by $f^{-1/2}$ or $f^{-3/2}$, and transformed back to the time domain. This is a special case of a general method for generating colored noise; the author has no citation for its actual use to generate flicker FM, but Ref. 3 of [2] cites a suggestion for its use.

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- Impulse response (IR) generators, discrete-time analogs of Riemann–Liouville fractional integration (see [2] for continuous-time power-law noise models). White noise is convolved with a causal filter that represents summation of order 1/2 or 3/2. The Kasdin–Walter algorithm [3, 4], which is in current use, performs the discrete convolution in $N \log N$ time by using the FFT.

Our principal aim here is to compare a DS generator, an IR generator, and two new FFT-based flicker FM generators that use the recent method of circulant embedding for exact simulation of stationary processes [5, 6, 7, 8]. Two properties are used for the comparison: 1) Allan deviation; 2) a form of mean square time interval error with a straight line removed. Lest this work be merely a discussion of models and algorithms, we also compare the generator outputs to phase residuals of two precision quartz oscillators that were chosen for flatness of their Allan deviations. We shall expose a deficiency of the IR generator, the same deficiency that a Barnes–Jarvis generator has if it is not properly initialized [9]: the generator’s long-term phase excursions are too small on average. Nevertheless, as Schmidt also found [10], one can work around this deficiency by generating twice as many points as needed and using only the second half of the output.

2 Two flicker FM phase models

Before defining the four generators, we define two flicker FM models, against which the generators can be compared. Each model is a discrete-time stochastic process x_n with sample period 1, normalized so that its *two-sided* spectral density $S_x(f)$, $|f| \leq 1/2$, is asymptotic to $|2\pi f|^{-3}$ as $f \rightarrow 0$. In the Conclusions, we give the formula for scaling these normalized models and generators to agree with the conventions used in time and frequency. Although x_n is nonstationary, its second increment $\Delta^2 x_n = x_n - 2x_{n-1} + x_{n-2}$ is a stationary, mean-zero, Gaussian process. The definition of x_n is ambiguous in that any constant phase and frequency can be added to it; to specify x_n exactly, we can fix two values x_a and x_b . The two models differ mainly in their spectral densities near the Nyquist frequency $f = 1/2$ (see Fig. 2).

2.1 FD(3/2) (fractional difference) model

For each value of the real parameter δ , there is a process called FD(δ) [11, 12] with spectral density $|2 \sin \pi f|^{-2\delta}$ (in a sense to be explained for $\delta = 3/2$). This family has the convenient property that if x_n is an FD(δ), then Δx_n is an FD($\delta - 1$). (The frequency response of the difference operator Δ is $|2 \sin \pi f|$.) In particular, FD($-1/2$) is defined as a stationary, mean-zero, Gaussian process z_n with spectral density $|2 \sin \pi f|$. For its autocovariance (ACV) sequence, $s_{z,n} = \text{E}z_j z_{j+n}$, we have

$$s_{z,n} = \int_0^1 e^{i2\pi f n} (2 \sin \pi f) df = \left[\pi \left(\frac{1}{4} - n^2 \right) \right]^{-1}. \quad (1)$$

By definition, x_n is an FD(3/2) process if $\Delta^2 x_n$ is an FD($-1/2$) process. Then x_n is a process with stationary second increments, and $S_x(f) = |2 \sin \pi f|^{-3}$ in the following sense: if H is a finite moving-average filter that contains Δ^2 as a factor, then Hx_n is stationary, and $S_{Hx}(f) = |H(e^{-i2\pi f})|^2 S_x(f)$, where $H(z)$ is the z -transform of H .

It can be proved that an FD(3/2) process x_n has the following representation:

$$\begin{aligned} x_n &= \left(1 + \frac{n}{n_1}\right) x_0 + \frac{n}{n_1} x_{-n_1} \\ &= \sum_{j=1}^n a_{n-j} u_j + \sum_{j=-\infty}^0 \left[a_{n-j} - \left(1 + \frac{n}{n_1}\right) a_{-j} + \frac{n}{n_1} a_{-n_1-j} \right] u_j, \end{aligned} \quad (2)$$

where n_1 is any positive integer, u_n is a standard white noise sequence (independent Gaussians with mean 0 and variance 1), and a_n is defined by the power series $(1-z)^{-3/2} = \sum_{n=0}^{\infty} a_n z^n$; thus $a_n = (-1)^n \binom{-3/2}{n}$, and $a_n = 0$ for $n < 0$ by convention. One can show that $a_n \sim n^{1/2}/\Gamma(3/2)$ as $n \rightarrow \infty$.

2.2 Sampled PPL (pure power law) model

Starting with a continuous-time process $x(t)$ with stationary second increments and spectral density $|2\pi f|^{-3}$ for all real nonzero f , we sample it at the integers to get a discrete-time process $x(n)$. Its second increment, $z(n) = \Delta^2 x(n)$, is stationary and has ACV

$$s_z(n) = s_x(n+2) - 4s_x(n+1) + 6s_x(n) - 4s_x(n-1) + s_x(n-2), \quad (3)$$

where $s_x(t)$ is the generalized ACV [13, 14] of $x(t)$:

$$s_x(t) = \frac{1}{2\pi} t^2 \ln |t| \quad \text{for } t \neq 0, \quad s_x(0) = 0. \quad (4)$$

The spectral density of the sampled process is not $|2\pi f|^{-3}$, but

$$\sum_{k=-\infty}^{\infty} |2\pi(f+k)|^{-3}, \quad |f| \leq 1/2. \quad (5)$$

3 Two general algorithms

Three of the generators under discussion can be quickly given in terms of two algorithms of wider utility; the second algorithm uses the first. With slight modifications, these descriptions follow Percival and Walden [5]. Although we use a complex FFT, an equivalent real version can also be used.

3.1 Discrete spectrum algorithm

Purpose: Generate values of a real stationary Gaussian process with a desired discrete spectrum.

Inputs: N (a power of 2), nonnegative numbers S_0, S_1, \dots, S_N , where S_k is the desired two-sided spectral density at frequency $f_k = k/(2N)$.

Outputs: Random variables z_0, z_1, \dots, z_N such that

$$\mathbb{E} z_m z_n = \frac{1}{2N} \sum_{k=1-N}^N S_{|k|} \exp(i2\pi f_k (n-m)).$$

Procedure:

Generate $U_0, U_1, \dots, U_N, V_1, \dots, V_{N-1}$ as independent standard Gaussians.

Let $Z_0 = \sqrt{S_0}U_0$, $Z_N = \sqrt{S_N}U_N$.

Let $Z_k = \sqrt{\tilde{S}_k}/2(U_k + iV_k)$, $Z_{2N-k} = Z_k^*$ for $k = 1$ to $N - 1$.

Generate z_0, \dots, z_{2N-1} (real-valued) as $\sqrt{2N}$ times the inverse FFT of Z_0, \dots, Z_{2N-1} . In other words,

$$z_n = (2N)^{-1/2} \sum_{k=0}^{2N-1} Z_k \exp(i2\pi f_k n).$$

Keep the values z_0, \dots, z_N (or any $N + 1$ consecutive values).

Remark: The full $2N$ -vector z_n is a $2N$ -periodic stationary process with spectrum S_k ; this, however, is not usually what one wants.

3.2 Circulant embedding algorithm

Purpose: Generate values of a real stationary Gaussian process with a given autocovariance.

Inputs: N (a power of 2), real numbers s_0, \dots, s_N , the desired autocovariance up to lag N .

Outputs: Random variables z_0, z_1, \dots, z_N such that $Ez_m z_n = s_{n-m}$ for $0 \leq m \leq n \leq N$.

Procedure:

Let $\tilde{s}_n = s_n$ for $n = 0$ to N , $\tilde{s}_{2N-n} = s_n$ for $n = 1$ to $N - 1$ (even circular extension of s_n).

Remark: "Circulant" refers to the covariance matrix that corresponds to \tilde{s}_n .

Let $\tilde{S}_0, \dots, \tilde{S}_{2N-1}$ be the FFT of $\tilde{s}_0, \dots, \tilde{s}_{2N-1}$. (Then \tilde{S}_k is real-valued.)

If any $\tilde{S}_k < 0$, the method *fails*. (This means that the extended circular sequence is not positive definite.)

Use $\tilde{S}_0, \dots, \tilde{S}_N$ in the discrete spectrum algorithm to generate the z_n .

Remark: The \tilde{S}_k are an artificial construct of the algorithm.

4 Four flicker FM generators

All these generators produce approximately N phase values x_n , where N is a power of 2. The first two are approximate; they do not simulate either target model exactly. The last two give exact simulations of the two target models.

4.1 DS - discrete spectrum

Let $f_k = k/(2N)$. Run the discrete spectrum algorithm with input $S_0 = 0$, $S_k = (2\pi f_k)^{-3}$, $k = 1, \dots, N$, and output x_0, \dots, x_N . (One could also use $S_k = (2 \sin \pi f_k)^{-3}$ to approximate the FD(3/2) model more closely.)

4.2 IR - impulse response

This generator is an approximate simulation method for the FD(3/2) model. Its output is given by the formula

$$x_n = \sum_{j=1}^n a_{n-j} u_j, \quad n = 1, \dots, N, \quad (6)$$

where a_n is defined after (2), and u_1, \dots, u_N are independent standard Gaussians. This convolution is carried out by zero-padding the sequences to length $2N$, Fourier transforming them, multiplying the transformed sequences, and inverse transforming the result. For details, see [3]. Observe that (6) is just one part of (2).

4.3 FD - fractional difference

This generator is an exact simulation of $N + 3$ values x_n of the FD(3/2) model. Run the circulant embedding algorithm using the ACV (1) for the input s_0, \dots, s_N , and z_0, \dots, z_N as output. It can be proved that the algorithm succeeds (the ACV satisfies Craigmile's criterion [8]). This produces an exact realization of $N + 1$ values of FD(-1/2). Then perform two cumulative summations:

$$\begin{aligned} y_n &= y_{n-1} + z_{n-1} \quad \text{for } n = 1 \text{ to } N + 1, \\ x_n &= x_{n-1} + y_{n-1} \quad \text{for } n = 1 \text{ to } N + 2. \end{aligned}$$

The initial values y_0 and x_0 are arbitrary, and may be set to zero.

4.4 PPL - pure power law

This generator is an exact simulation of $N + 3$ values of the sampled PPL model. It is identical to the FD generator just described except that the ACV (3) is used in place of (1). Again, it can be proved that the algorithm succeeds. There is one complication: to avoid catastrophic roundoff error in (3), use the asymptotic approximation

$$s_z(n) \approx -\frac{1}{\pi n^2} \left(1 + \frac{1}{n^2} + \frac{3}{2n^4} \right) \quad (7)$$

in place of (3) whenever $n \geq 35$.

5 Comparisons

We compare the four flicker FM generators with each other and with the phase residuals of two quartz oscillators (Oscilloquartz and CMAC) that were compared once per second against hydrogen masers. The test runs were chosen for flatness of Allan deviation between 1 and 1000 seconds¹.

Figure 1 shows a sample output of the four generators with $N = 1024$. Also shown are the first 1025 phase residuals of the quartz oscillators, scaled up as explained in the section below on Allan deviation. The exact generators are both initialized so that $x_0 = x_1 = 0$. The IR output starts with a small nonzero value of x_1 . The DS output, which is a sample of a stationary process, has no special initial value.

Figure 2 shows the spectral density of the two target models along with the discrete spectrum of the DS generator for $N = 32$. The spectral densities, multiplied by $(2\pi f)^3$, are plotted on a linear scale against f . As the next section shows, the rise in the sampled PPL spectrum (5) near the Nyquist frequency is just right to make its Allan deviation exactly flat for all integral τ . The DS spectrum actually has too little high-frequency power for this purpose, the FD spectrum too much. For most purposes, though, these high-frequency deviations are insignificant.

5.1 Allan deviation

Figure 3 shows the theoretical Allan deviation (lines) and the square root of measured Allan variance (small dots), averaged over 10000 trials, for the four generators with $N = 1024$. Also shown are the measured Allan deviations (symbols) of the quartz oscillators, normalized so that $\sigma_y(64\text{s}) = \sqrt{(\ln 4)/\pi} \doteq 0.664$, the theoretical value assumed by the PPL model for all τ . The σ axis has an expanded linear scale to bring out the differences among the plots. The IR generation was actually

¹Thanks to Al Kirk for making these tests available.

performed with $N = 2048$; IR1 refers to the first half of the generated sequence, which is equivalent to IR with $N = 1024$; IR2 refers to the second half. We see the expected minor deviations from flatness for small τ , and an insignificant droop by DS and IR1 at $\tau = 512$. The PPL line is exactly flat. The Allan variance indicates little difference among the generators.

5.2 Two-point MSTIE

Suppose that the phase $x(t)$ of a clock is measured at times $t_0 - \tau_1$ and t_0 . For the purpose of this discussion, the mean square time interval error of a clock (MSTIE) after a delay τ is defined by

$$\text{MSTIE}(\tau, \tau_1) = \text{E} \left[x(t_0 + \tau) - \left(1 + \frac{\tau}{\tau_1} \right) x(t_0) + \frac{\tau}{\tau_1} x(t_0 - \tau_1) \right]^2, \quad (8)$$

which is the mean square error of linear extrapolation from the two phase measurements. We assume that it is independent of t_0 ; this is so for all processes $x(t)$ with stationary second increments. Although the method of phase calibration is crude, this measure serves the purpose of showing the variance of the long-term phase deviations as we go farther and farther from a fixed calibration interval. (See [15] for a discussion of more sophisticated calibrations.) Figure 4 plots theoretical and average measured values of $\text{MSTIE}(\tau, \tau_1) / \tau^2$ against τ with $\tau_1 = 10$ for the flicker FM generators and the quartz oscillators, normalized as before. The MSTIE for the oscillators was measured by averaging the squared extrapolation error over t_0 with τ and τ_1 fixed; the averaging time was about 45000 s for the Oscilloquartz, 129000 s for the CMAC. All the curves except IR1 show the same asymptotic $\pi^{-1} [1 + \ln(\tau/\tau_1)]$ behavior that is calculated for the PPL model [9], with a tiny droop at the largest τ for the DS generator. The IR2 curve cannot be distinguished from the FD curve, but the IR1 curve droops significantly as τ increases.

6 Conclusions

- To simulate samples $x(n\tau_0)$ of flicker FM phase (time) $x(t)$ with one-sided frequency spectrum $S_y^+(f) = h_{-1}f^{-1}$, Allan deviation $\sqrt{h_{-1} \ln 4}$, multiply the output x_n of a normalized flicker FM generator by $\sqrt{\pi h_{-1} \tau_0}$.
- All these flicker FM generators take roughly the same programming effort and running time.
- The FD and PPL generators give exact simulations of their nonstationary target models. Even on a finite interval, their outputs are affected by arbitrarily low Fourier frequencies. The DS generator output, though it is a stationary process, still behaves substantially like a nonstationary f^{-3} process on a finite time interval.
- The DS, FD, and PPL generators behave more like a quartz oscillator than the IR generator does. Although the Allan deviation of both oscillators rises as τ increases beyond 64 s, this rise is not enough to explain why the oscillator MSTIE points line up with the curves for DS, FD, PPL, and IR2 (the “second-half” modification of IR), but not with the IR1 curve.
- The IR generator is an inaccurate simulator of the FD(3/2) model; it has a “burn-in” problem, which causes it to have smaller long-term phase deviations on average than the model does. Fortunately, one can still get an accurate FD(3/2) simulation from IR by generating twice as many points as one needs and throwing out the first half. In this case, one has to use FFTs of size $4N$ to generate N points. One can also use an FFT size of $4N$ in the N -point DS generator to fill in more low frequencies.

- The IR generator is deficient because it neglects the past of the FD(3/2) process; the same is true for the fractional integral models in [2]. The IR output (6) is just the first term of the right side of the FD(3/2) formula (2), whose other terms represent the effect of the entire past ($j \leq 0$) on the future value x_n . In fact, the IR output is exactly the *error* of the mean-square optimal linear predictor of x_n on the past. If we could express the second sum in (2) in terms of x_j , $j \leq 0$, instead of u_j , then we would have the predictor itself. Similarly, if a Barnes–Jarvis generator starts with a zero initial state, then its output is a prediction error, not the whole target process [9]. It is one thing to tie the present to zero, as we often do; it is another thing to neglect the past of these long-memory processes.
- Flatness of Allan variance is an inadequate way to judge a flicker FM generator; it gives scarcely any hint of the deficiency of the IR generator.
- Although these models and generators (except IR) seem to behave in the right way, they probably give little insight into how $1/f$ noise arises in the world. It is as though we did not know that a process with a $1/f^2$ spectrum, which we could simulate approximately by the discrete spectrum algorithm with $S_k = f_k^{-1}$, is actually a random walk, the time integral of a sequence of independent random shocks.

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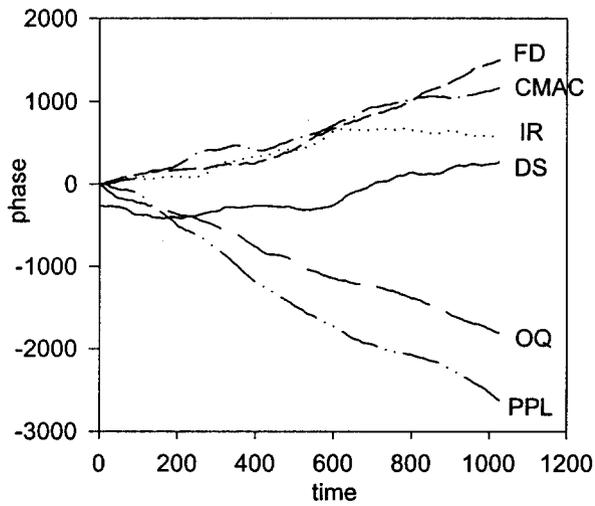


Fig. 1. Phase samples from flicker FM generators and quartz oscillators. Generators: DS = discrete spectrum, IR = impulse response, FD = fractional difference, PPL = pure power law. Oscillators (normalized): OQ = Oscilloquartz, CMAC.

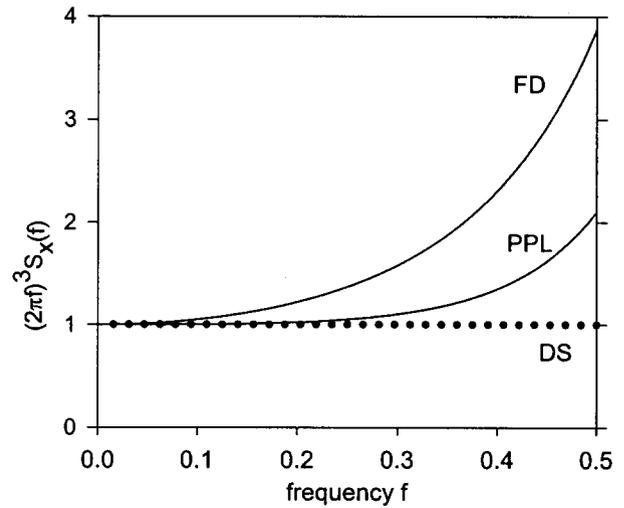


Fig. 2. Spectra of the DS generator ($N = 32$), the FD(3/2) model, and the sampled PPL model. The spectral densities are multiplied by $(2\pi f)^3$.

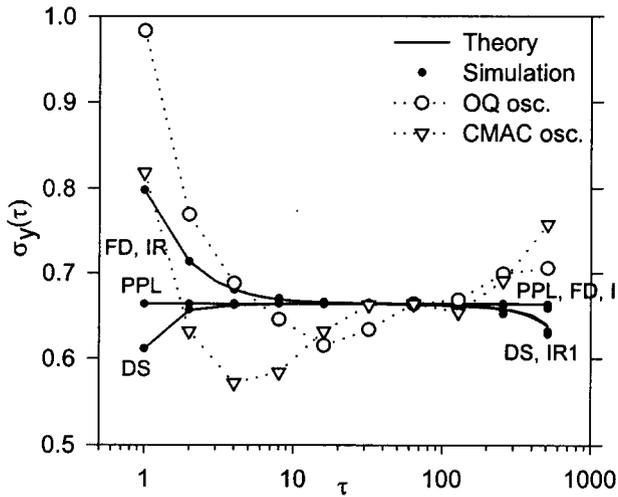


Fig. 3. Allan deviation of flicker FM generators and quartz oscillators (normalized). IR1 = first half of IR output with $N = 2048$, IR2 = second half.

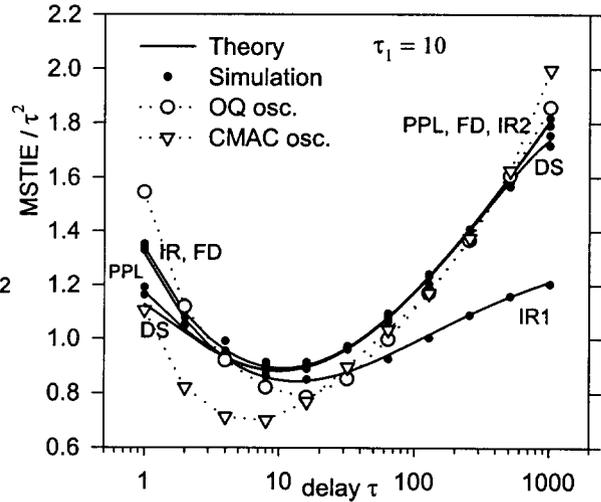


Fig. 4. Mean square time interval error, divided by τ^2 , for delay τ and calibration period $\tau_1 = 10$. All the flicker FM generators except IR1 behave like the normalized oscillators for large τ .

FFT-Based Methods for Simulating Flicker FM

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Outline

Motivation: Simulate flicker FM component of quartz oscillator noise

Noise model

Two software noise generators

- Old - approximate simulation of model

- New - exact simulation of model

Oscillator data (Oscilloquartz 8607-MB vs H-maser)

Comparison methods

- Allan deviation

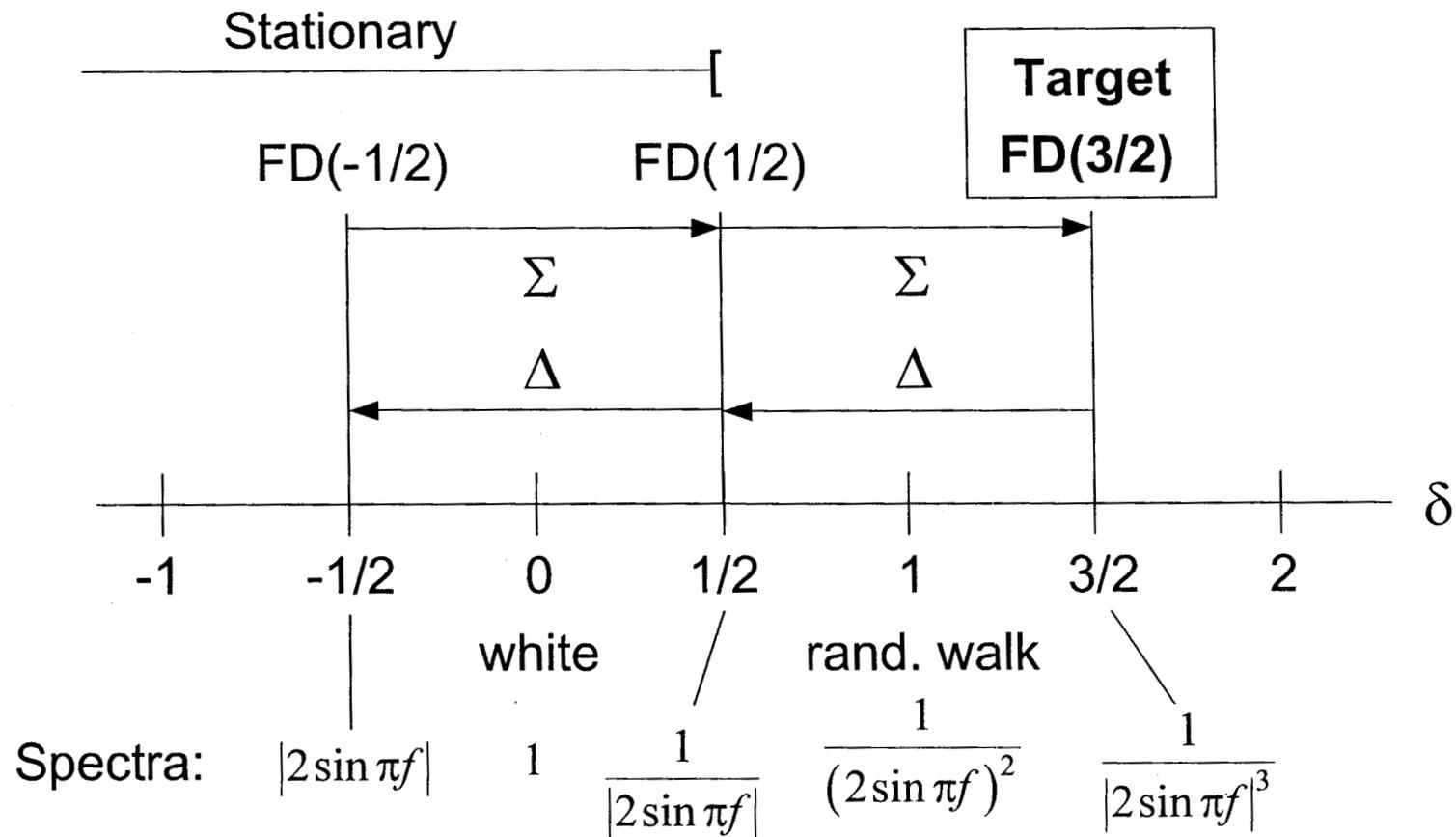
- Mean square time interval error (MSTIE)

Conclusions

Fractionally Differenced Processes

Process $FD(\delta)$ has spectrum $\frac{1}{|2 \sin \pi f|^{2\delta}}$

$$\Delta FD(\delta) = FD(\delta-1)$$



Impulse Response (IR) Generator

Kasdin and Walter, 1992

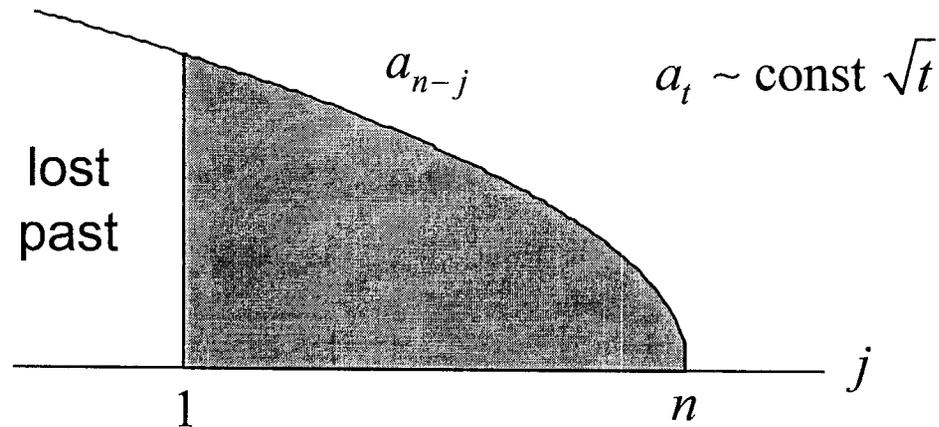
Approximate simulation of FD(3/2) model

Fractional integration filter of order 3/2:

$$A(z) = (1 - z^{-1})^{-3/2} = \sum_{n=0}^{\infty} a_n z^{-n}$$

Let $x_n = \sum_{j=1}^n a_{n-j} u_j$, $n = 1, 2, \dots, N$, where u_j is white noise.

Done in $N \log N$ time by FFT ($N = \text{power of } 2$)



FD Generator

Exact simulation of FD(3/2) model

1. Let $N =$ power of 2.
2. Generate z_0, \dots, z_N , $N + 1$ values of stationary FD(-1/2),
by exact method of **circulant embedding**.
3. Do two cumulative sums:

$$y_n = y_0 + \sum_{j=1}^n z_{j-1}, \quad x_n = x_0 + \sum_{j=1}^n y_{j-1}$$

$(x_0, y_0$ arbitrary)

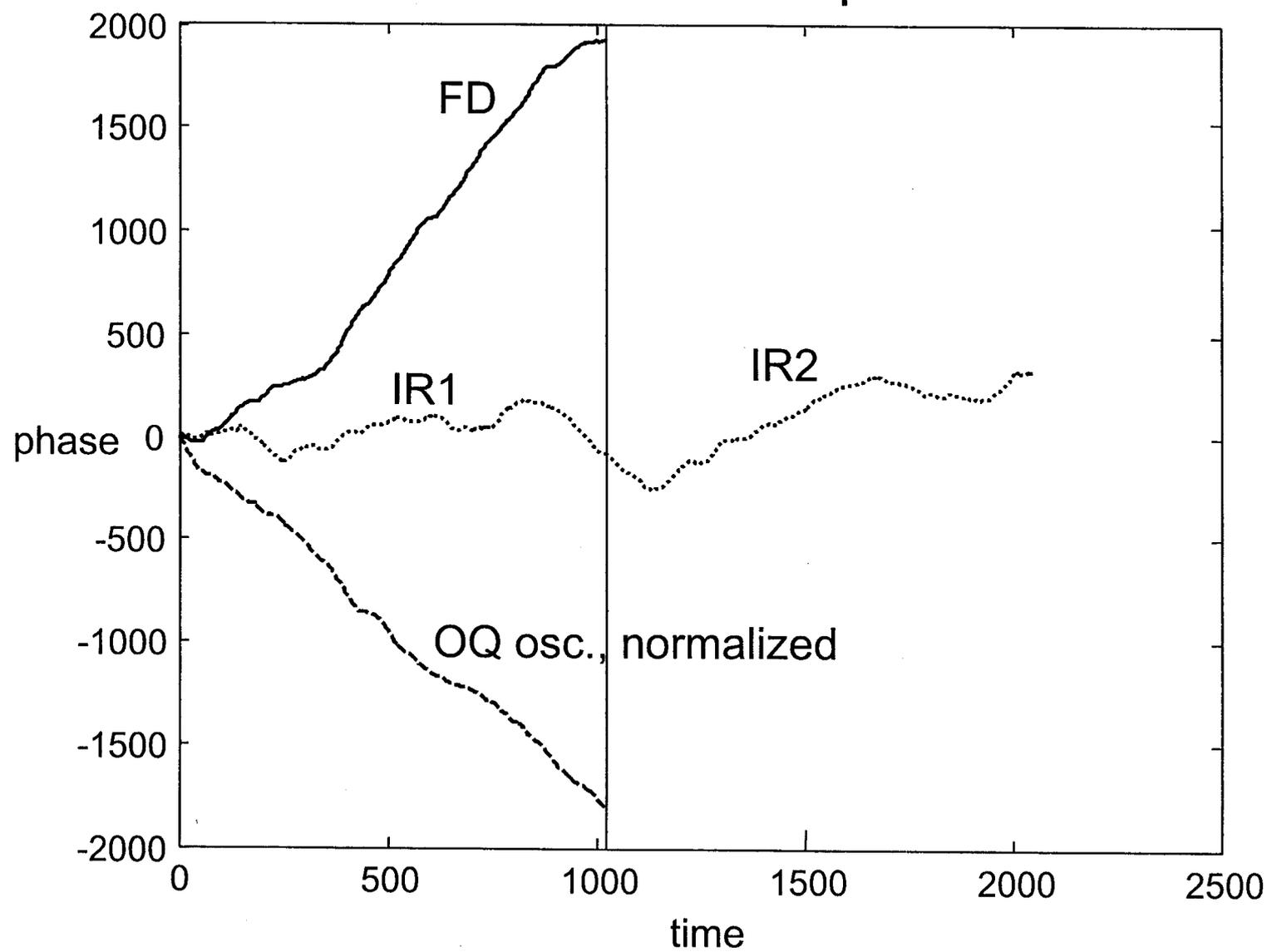
Then x_0, \dots, x_{N+2} is exact FD(3/2).

Circulant Embedding

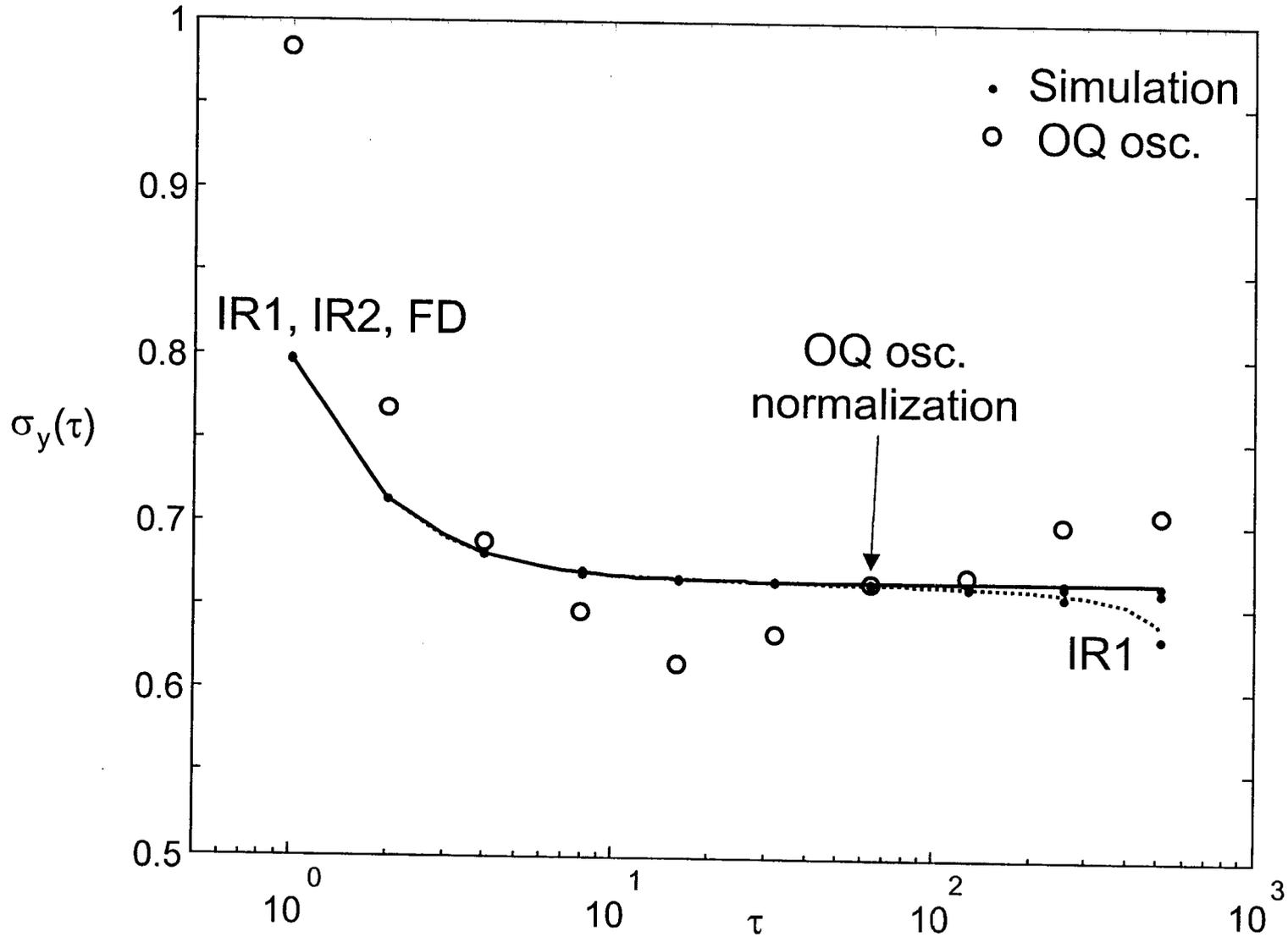
Exact method for simulating stationary Gaussian process with given autocovariance sequence

1. Let $N =$ power of 2.
2. Start from desired ACV s_0, \dots, s_N , where $s_t = E x_n x_{n+t}$
3. Let $\tilde{s} = [s_0, s_1, \dots, s_N, s_{N-1}, \dots, s_1]$, extended ACV
4. Let $\tilde{s} \rightarrow \text{FFT}_{2N} \rightarrow \tilde{S}$
5. If any $\tilde{S}_k < 0$, the algorithm **fails**.
6. Generate z_0, \dots, z_{2N-1} from discrete spectrum $\tilde{S}_0, \dots, \tilde{S}_{2N-1}$
7. Use only z_0, \dots, z_N

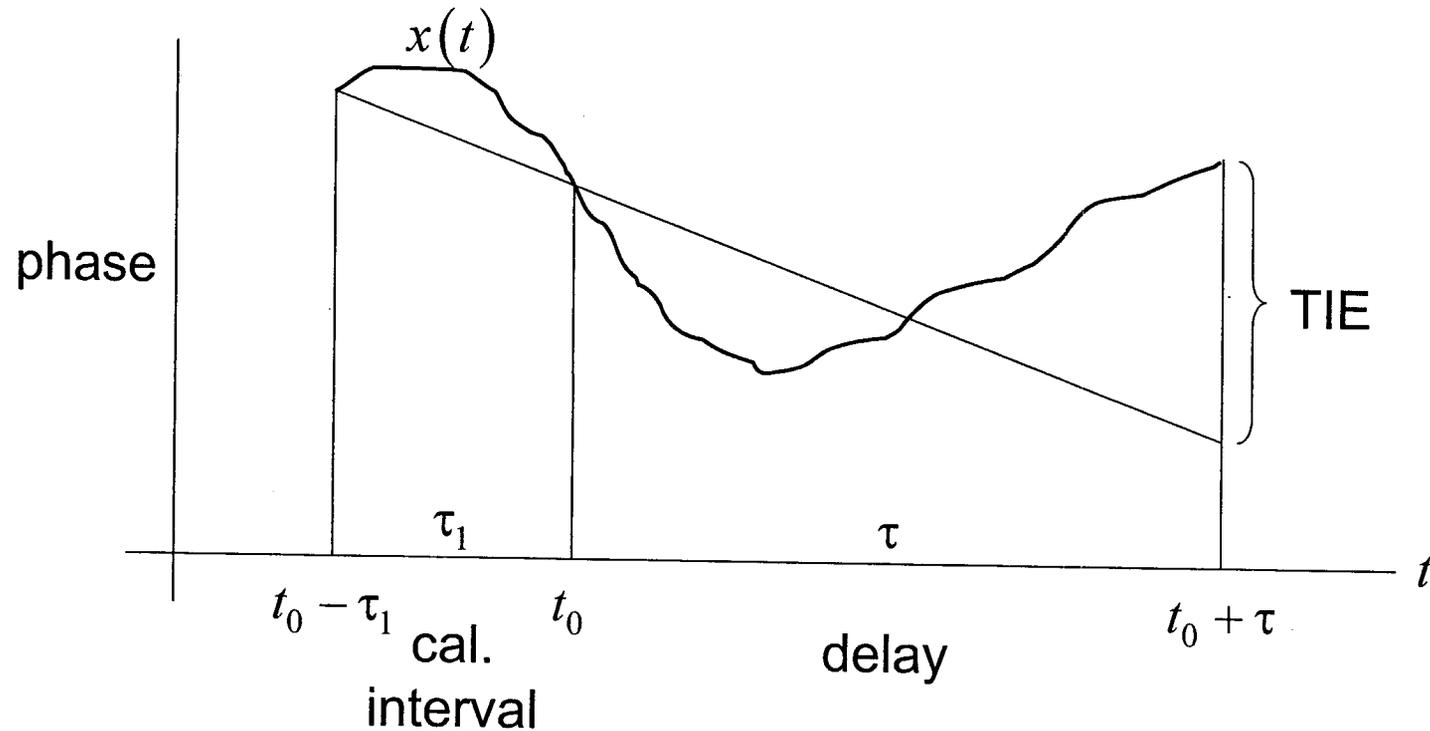
Flicker FM Samples



Allan Deviation Comparison



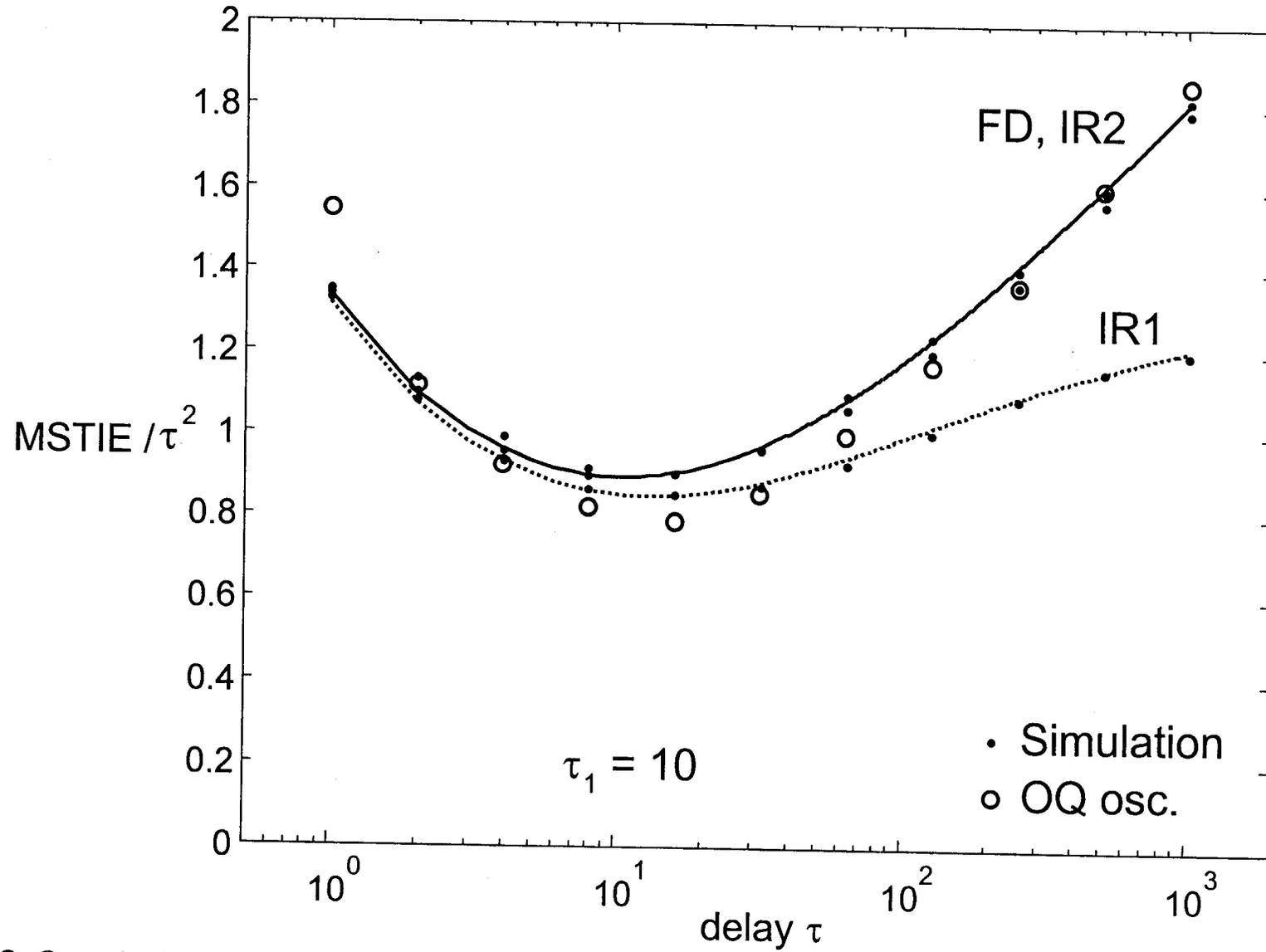
2-Point Time Interval Error (TIE)



Mean square TIE, MSTIE (τ, τ_1) :

Average TIE^2 over ensemble or time t_0 keeping τ_1 and τ fixed.

MSTIE Comparison



Conclusions

The FD generator gives an exact simulation of N points of the FD(3/2) flicker FM model.

The IR generator has a “burn-in” problem:

It has smaller long-term phase deviations on average than the model or a quartz oscillator.

Second-half workaround: before generating N points, develop a past of length N.

Flatness of Allan deviation is an inadequate criterion for judging flicker FM generators.

Parallel to 1986 work on Barnes-Jarvis generators.